Existence and approximation results of nonlinear hybrid functional differential equation of fractional order

Dr. Pravin M. More
Department of mathematics B.S.S.ASC College Makani, Maharashtra India

Abstract: In this paper we prove that the existence and approximation results for nonlinear functional hybrid differential equations are involving the Caputo fractional derivative order $0 < \alpha < 1$ using the hybrid fixed point theorem by using Dhage theorem.

Keywords Hybrid functional differential equation, Dhages theorem Caputo fractional derivatives approximation solutions

1.Introduction: Fractional calculus is generalization of ordinary differential equation and integration to arbitrary integer [15,17]. They have a played an important role in development of fractional calculus as a pure theoretical field ,useful only mathematicians [19].In last few decades fractional differential equation are useful in various field of applied science and engineering in physics ,chemistry, biology, control theory, electrochemistry, viscoelasticity, feedback amplifier and electric circuit[14,16,18,19].fractional differential equation have to collect something significant intersect in recent year due to rapidly development of fractional calculus across the various field. Recently Dhage[1-8] and Jadhav[3,4] ,Lu[9,11],Herzallah and Beleanu[12,13] are studied the quadratic perturbation of nonlinear differential equations has captured much attention. They have importance of the investigation of hybrid differential equations [10]. The existence solution has generalized to the fractional order hybrid functional differential equation.

$$\frac{d^\alpha}{dt^\alpha}[x(t) - f(x(\alpha(t))] = g(t, x(\gamma(t)) \text{ a.e. } t \,\epsilon [t_0, t_0 + a],$$

$$x(t_0) = x_0, \tag{1.1}$$

where for all $f \,\epsilon\, \mathfrak{I} \times \mathfrak{R} \to \mathfrak{R}\,,.\, g\colon \mathfrak{I} \times \mathfrak{R} \to \mathfrak{R}$ are continuous function where $\mathfrak{I} = [t_0, t_0 + a]$ is bounded interval in $\mathfrak{R}$ for some t_0 and $a \,\epsilon\, \mathfrak{R}$ with $a > 0, \mathbb{C}(\mathfrak{I} \times \mathfrak{R}, \mathfrak{R})$ is the class of continuous function.

In this paper we study the existence and approximation of solutions to using Caputo fractional order hybrid functional differential equations by using the Dhage's theorem.

In next section, we introduced some basic definitions, theorem and auxiliary results are used in subsequent section of the paper lastly, we prove the main result.

2. Preliminaries: Let $\mathfrak{I} = [t_0, t_0 + a]$ be a closed bounded interval in $\mathfrak{R}$ where $t_0 \geq 0, a > 0$.we denote the function $\mathbb{C}(\mathfrak{I}, \mathfrak{R})$ for the space of continuous functions $x\colon \mathfrak{I} \to \mathfrak{R}$. The space $\mathbb{C}(\mathfrak{I}, \mathfrak{R})$ in Banach space with the supremum norm$\|.\|$ is given by,

$$\|x\| = \sup_{t \epsilon J} |x(t)| \text{ for } x \epsilon\, \mathbb{C}(\mathfrak{I}, \mathfrak{R}) \tag{2.1}$$

$$\text{and } x < y \Leftrightarrow x(\alpha(t)) \leq y(\alpha(t)) \tag{2.2}$$

let $(\mathbb{E}, \preccurlyeq . \|.\|)$ denote a partially ordered normed linear space. Two elements x and y in $\mathbb{E}$ are said to be the comparable if either relation $x \preccurlyeq y$ or $y \preccurlyeq x$ holds. A nonempty subset $\mathbb{C}$ of $\mathbb{E}$ is called chain or totally ordered if all the element of $\mathbb{C}$ ae comparable then $\mathbb{E}$ is known as $\mathbb{E}$ is said to regular if for any nondecreasing sequence$\{x_n\}_{n \epsilon \mathbb{N}}$ in $\mathbb{E}$ such that $x_n \to x^* \; as \; n \longrightarrow \infty \; then \; x_n \preccurlyeq x^* \; or x_n \succcurlyeq x^*$ for all $n \epsilon \mathbb{N}$

particular the space is $\mathbb{C}(\mathfrak{I},\mathfrak{R})$. The regularity in $\mathbb{E}$ may be found Guo and Lakshmikanthan [21,22] and there references their function in[10,16,18,20].

Definition2.1: [7] A mapping $\mathbb{T}:\mathbb{E}\to\mathbb{E}$ is called the nondecreasing if the order relation is preserved under $\mathbb{T}$, that is for any $x,y\,\epsilon\,\mathbb{E}$ such that $x\le y, then\ we\ have\ \mathbb{T}x\le\mathbb{T}y.$

Definition2.2[7,8] An upper semi-continuous and monotone nondecreasing function $\psi:\mathfrak{R}_+\to\mathfrak{R}_+$ is called $\mathbb{D}$-function provided $\psi(0)=0$.An operator $\mathbb{T}:\mathbb{E}\to\mathbb{E}$ is called partially nonlinear $\mathbb{D}$-contraction if there exist $\mathbb{D}$-function ψ such that,

$$\|\mathbb{T}x-\mathbb{T}y\|\le\psi(\|x-y\|) \qquad (2.3)$$

For all comparable elements $x,y\in\mathbb{E}, where\ 0<\psi(\mathrm{r})<\mathrm{r}$ for $\mathrm{r}>0$.

In particular, if $\psi(\mathrm{r})=kr, k>0$, $\mathbb{T}$ is called a partial Lipschitz operator with Lipschitz constant k and moreover, $0<k<1$, $\mathbb{T}$ is called partial linear contraction on $\mathbb{E}$ with contraction with a contraction constant $\mathbb{K}$. Moreover $\mathbb{T}$ is said to be nonlinear $\mathbb{D}$ -contraction if it is nonlinear $\mathbb{D}$ -Lipschitz with $\psi(\mathrm{j})<\mathrm{j}$ forall $\mathrm{j}>0$.

Definition 2.4[7,8]: Let $\mathbb{E}$ be nonempty set equipped with an order relation $\preccurlyeq$ *and* a metric $\mathbb{D}$. we say that the order relation $\preccurlyeq$ and the metric $\mathbb{D}$ are compatible if the following property ,is satisfied: if $\{x_n\}_{n\in\mathbb{N}}$ is a monotone sequence in $\mathbb{E}$ for which a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*,then whole sequence $\{x_n\}_{n\in\mathbb{N}}$converges to x^*.Similarly if $(\mathbb{E},\preccurlyeq,\|.\|)$ Is partially ordered linear space, we say that the order relation $\preccurlyeq$ and the $\|.\|$ are compatible whenever the order relation $\preccurlyeq$ and the metric d induced by the norm $\|.\|$ are compatible.

Definition2.5[7,8] An operator $\mathbb{T}:\mathbb{E}\to\mathbb{E}$ is called partially compact $a\epsilon\mathbb{E}$,the set $\mathbb{C}$ in $\mathbb{E}$ the set $\mathbb{T}(\mathbb{C})$ is relatively compact subset of $\mathbb{E}$.an operator $\mathbb{T}$ is said to be partially totally bounded if for ant totally ordered and bounded subset $\mathbb{C}$ of $\mathbb{E}$ the set $\mathbb{T}(\mathbb{C}\ )$ is relatively compact subset of $\mathbb{E}$. If $\mathbb{T}$ is partially continuous and partially totally bounded, then the partially completely continuous operator in $\mathbb{E}$.

Definition2.6[14,15,16] for $f(t)\epsilon\,\mathbb{L}(a,b)$ we see all integrable functions and $\alpha>0$ then the left Riemann-Liouville fractional integral order α is defined by,

$$I^\alpha f(t)=\frac{1}{\Gamma(\alpha)}D^n\int_a^t(t-\tau)^{\alpha-1}f(\tau)d\tau \qquad (2.4)$$

$$where\ n\ is\ such\ that\ n-1<\alpha<n\ and\ D=\frac{d}{dt.}$$

Definition 2.7[14,15,16] For $\alpha>0$ the left Caputo fractional derivatives order α is defined by,

$$D^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t(t-\tau)^{n-\alpha-1}f(\tau)d\tau \qquad (2.5)$$

$$where\ n\ is\ such\ that\ n-1<\alpha<n\ and\ D=\frac{d}{d\tau.}$$

Definition 2.8[14,15,16] The function $x\epsilon\mathbb{C}(\mathfrak{I},\mathfrak{R})$ is called the mild solution of the fractional nonlinear hybrid functional differential equation (1.1) is satisfies the integral equation,

$$x(t)=f(t,x(\alpha(t))+x_0-f(t_0-x_0)+\int_{t_o}^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g\left(s,x\big(\gamma(s)\big)\right),ds\ \ t\in\mathfrak{I} \qquad (2.6)$$

We remark that the mild solution is given by (2.6) is obtained by from (1.1) by applying the fractional integral I^α defined by,

$$I^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau f(\tau)d\tau$$

to both sides.

The following hybrid fixed point result [5,7,12] is often applied the existence and approximation of solution of various differential and integral equations.

Theorem 2.1([8,12]) Let $(\mathbb{E}, \preccurlyeq, \|.\|)$be a regular partially ordered complete normed linear space such that the every compact chain $\mathbb{C}$ in E. let $A, \mathbb{B}: \mathbb{E} \to \mathbb{E}$ tobe nondecreasing operator such that,

a) $\mathbb{A}$ is partially nonlinear $\mathbb{D}$-contraction.
b) $\mathbb{B}$ is partially continuous and partially compact.
c) Ther exists an element $\alpha_0 \in \mathbb{X}$ $such\ that,$

$$\alpha_0 \preccurlyeq \mathbb{A}_{\alpha_0} + B_{\alpha_0}\ or\ \alpha_0 \succcurlyeq \mathbb{A}_{\alpha_0} + \mathbb{B}_{\alpha_0}.$$

Then the operator equation $\mathbb{A}x + \mathbb{B}x = x$ has a solution x^* and the sequence $\{x_n\}$ of successive iteration defined by, $x_0 = \alpha_0$ then

$x_{n+1} = \mathbb{A}_{x_n} + \mathbb{B}x_n$ $, n = 0,1,2,3,$ ...converges monotonically to x^*.

Remark 2.1: The condition that every compact chain $\mathbb{E}$ is holds if every partially compact subset of $\mathbb{E}$ possesses the combability property with respect to the order relation $\preccurlyeq$ and the norm $\|.\|$ in it. this simple fact used to prove the main results of the paper,

3.Existence theory

The function $x \in \mathbb{C}(\mathfrak{I}, \mathfrak{R})$is Banach space with respect to above supremum and also partially ordered w.r.t. the above partially ordered relation $\preccurlyeq$. It is known as partially ordered Banach space $\mathbb{C}(\mathfrak{I}, \mathfrak{R})$ is regular and lattice so that every pair of the element of $\mathbb{E}$ has a lower and an upper bound in it. See dhage [5,6,7] and their reference there in [18].

Lemma3.1 Let ($\mathbb{C}(\mathfrak{I}, \mathfrak{R}), \preccurlyeq, \|.\|$)be a partially ordered Banach space with norm $\|.\|$ and the order relation $\preccurlyeq$ is defined by (2.1) and (2.2) the respectively. Then every partially compact subset of $\mathbb{C}(\mathfrak{I}, \mathfrak{R})$

Proof: the proof of the lemma is well-known as appears in the papers of Dhage [3]. We introduce an order relation $\preccurlyeq c\ in\ \mathbb{C}$ induced by the order relation $\leq$ defined in $\mathbb{C}(\mathfrak{I}, \mathfrak{R})$. Thus, for any $x, y \in \mathbb{C}, x \leq cy$ it implies that $x(\alpha(\theta)) \leq y(\alpha(\theta))$ for all $\theta \in I_0$.Moreover if $x, y \in \mathbb{C}(\mathfrak{I}, \mathfrak{R})$and $x \leq y\ \ then\ x_t \leq cy(t)$ for all $t \in \mathbb{I}$.

Definition 3.1[8]. A function $x\epsilon\ \mathbb{C}(\mathfrak{I}, \mathfrak{R})$ is said to be solution of the HFDE (1.1) if

a)$x_0 \epsilon\ \mathbb{C}$.

b)$x(t)\ \epsilon\ \mathbb{C}$ for each $t\epsilon\ \mathbb{I}$,and

c) the function $t \mapsto [x(t) - x(\alpha(t))]$ is continuously differentiable on $\mathbb{I}$ satisfies the equation in (1.1),

we consider the following set of assumptions;

($\boldsymbol{H_1}$) The function $f: \mathfrak{I} \times \mathfrak{R} \to \mathfrak{R}$ and$g: \mathfrak{I} \times \mathfrak{R} \to \mathfrak{R}$ are continuous.

($\boldsymbol{H_2}$) f is nondecreasing in x for each $t \in \mathfrak{I}\ and\ x \in \mathfrak{R}$.

($\boldsymbol{H_3}$)There exists $\mathbb{M}_f > 0$ such that $0 \leq |f(t, x)| \leq \mathbb{M}_f$ for all $t\epsilon\ \mathfrak{I}\ \ and\ \ x \in \mathfrak{R}$.

($\boldsymbol{H_4}$)There exists a $\mathbb{D}$ -Contraction $\emptyset$ such that,

$$0 \leq |f(t, x) - f(t, y)| \leq \emptyset(x - y)\ for\ t \in \mathfrak{I} \text{ and } x, y \in \mathfrak{R}\ with\ x \geq y.$$

$(\boldsymbol{H_5})$ g is nondecreasing in x for $t \in \mathfrak{I}$ and $x, y \in \mathfrak{R}$.

$(\boldsymbol{H_6})$There exists constant $\mathbb{M}_g > 0$ such that $0 \leq |f(t,x)| \leq \mathbb{M}_g$for all $t\epsilon\, \mathfrak{I}$ and $x \in \mathfrak{R}$.

$(\boldsymbol{H_7})$ There exists function $u \in \mathbb{C}(\mathfrak{I}, \mathfrak{R})$ such that lower solution the problem (1.1) that is,

$\frac{d^\alpha}{dt^\alpha}[u(t) - f(t, u(\alpha(t))] \leq g\left(t, u\big(\gamma(t)\big)\right)$ $, t \in \mathfrak{I}$ and $t \in f(t, u(\alpha(t), u(t_0) \leq x \leq \mathfrak{R}$.

4.Main Result;

Theorem4.1: Suppose that the hypothesis $(H_1 - H_7)$ are satisfied then the initial value problem (1.1) has a mild solution $x^*, \mathfrak{I} \in \mathfrak{R}$ and sequence of successive approximation x_n where $n = 1,2,3, \dots ..$ defined by,

$$x_{n+1}(t) = f\left(t, x_n\big(\alpha(t)\big)\right) + x_0 - f(t_0 - x_0) + \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_n\big(s(\gamma)\big)\right) ds$$

$$x\big(\alpha(t)\big) = u\big(\alpha(t)\big),$$

Converges monotonically to x^*.

Proof: We take the partially ordered Banach space $\mathbb{E} = \mathbb{C}(\mathfrak{I}, \mathfrak{R})$.we prove the existence of solution (1.1) by considering the equivalence operator equation

$$\mathbb{A}x(t) + \mathbb{B}x(t) = x(t)$$

where $\mathbb{A}x(t) = \left(t, x\big(\alpha(t)\big)\right)$

$$\mathbb{B}x(t) = x_0 - f(t_0 - x_0) + \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x\big(\gamma(s)\big)\right) ds$$

for $t\epsilon\mathfrak{I}$ we shall to prove that $\mathbb{A}$ and $\mathbb{B}$ are satisfy all conditions in theorem 2.1

Step I: First of all we prove that $\mathbb{A}$ and $\mathbb{B}$ are nondecreasing operators. For any $x, y \in \mathbb{E}$ with $x \geq y$, we obtain from the assumption (H_3)

$$\mathbb{A}x(t) = f(t, x\big(\alpha(t)\big) \geq f(t, y\big(\alpha(t)\big) = \mathbb{A}y(t)$$

This meaning $\mathbb{A}$is nondecreasing. For $\mathbb{B}$ we have from the assumption (H_5)

$$\mathbb{B}x(t) - \mathbb{B}y(t) = \int_{0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left| g\left(s, x\big(\gamma(t)\big)\right) - g\left(s, y\big(\gamma(t)\big)\right) \right| ds$$
$$\geq 0$$

For any $x \geq y$ in $\mathbb{E}$ therefore the operator $\mathbb{B}$ is also nondecreasing.

Step II: We show that the operator $\mathbb{A}$ is satisfies the condition (a) in theorem 2.1 that is $\mathbb{A}$ is partially bounded and partially nonlinear $\mathbb{D}$-contraction on $\mathbb{E}$, for this purpose. Let $x\epsilon\mathbb{E}$ be arbitrary then by the bounded condition (H_3) we see that,

$$|\mathbb{A}x(t)| = \left| f\left(t, x\big(\alpha(t)\big)\right) \right| \leq \mathbb{M}_f$$

for all $t\epsilon\mathfrak{I}$ therefore $\|\mathbb{A}x\| \leq \mathbb{M}_f$, which shows that $\mathbb{A}$ is partially bounded in $\mathbb{E}$. Moreover, for any $x, y \in \mathbb{E}$ such that $x \geq y$ we see that assumption (H_4).

$$\left| ax\big(\alpha(t)\big) - ay\big(\alpha(t)\big) \right| = \left| f\left(t, x\big(\alpha(t)\big)\right) - f\left(t, y\big(\gamma(t)\big)\right) \right|$$

$$\leq \emptyset\left|x\big(\alpha(t)\big) - y\big(\gamma(t)\big)\right|$$
$$\leq \emptyset(\|x(t) - y(t)\|)$$
$$\leq \emptyset(\|x - y\|)$$

For $t\epsilon\mathfrak{J}$ where the inequalities are obtained the condition that $\emptyset$ is the nondecreasing hence $\|\mathbb{A}x - \mathbb{A}y\| \leq \emptyset(\|x - y\|)$ for all $x, y \in \mathbb{E}$ with $x \geq y$. This means $\mathbb{A}$ is partially nonlinear $\mathbb{D}$-contraction on $\mathbb{E}$ and thus partially continuous.

Step III: we prove that $\mathbb{B}$ is partially continuous and on $\mathbb{E}$.

Let $\{x_n\}_{n\in\mathbb{N}}$be the sequence in chain $\mathbb{E}$ are satisfying $x_n \to x \ as\, n \to \infty$.then we obtain from the boundedness of g in (H_6) is continuity of g in $(H_1)$and the dominated convergence theorem.

$$\lim_{n\to\infty}(\mathbb{B}x_n)(t) = \lim_{n\to\infty}\left(x_0 - f(t_0, x_0) + \int_{t_0}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g\left(s, x_n\big(\gamma(s)\big)\right)ds\right)$$
$$\leq \lim_{n\to\infty} x_0 - \lim_{n\to\infty} f(t_0, x_0) + \lim_{n\to\infty}\int_{t_0}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g\left(s, x_n\big(\gamma(s)\big)\right)ds$$
$$= x_0 - f(t_0, x_0) + \int_{t_0}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g\left(s, x_n\big(\gamma(s)\big)\right)ds$$
$$= (\mathbb{B}x)(t)$$

for each $t \in \mathfrak{J}$ this implies that $(\mathbb{B}x_n)$ converges to $(\mathbb{B}x)$ pointwise on $\mathfrak{J}$ and the convergence is monotonic by the property of g. Next, we show that $\{\mathbb{B}x_n\}_{n\in\mathbb{N}}$ is equicontinuous in $\mathbb{E}$.

Let $t_1, t_2 \in \mathfrak{J} = [t_0, t_0 + a]$ with $t_1 \leq t_2$. We have

$$|(\mathbb{B}x_n)(t_2) - (\mathbb{B}x_n)(t_1)|$$
$$= \left|\int_{t_0}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}g\left(s, x_n\big(\gamma(s)\big)\right)ds - \int_{t_0}^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}g\left(s, x_n\big(\gamma(s)\big)\right)ds\right|$$
$$\leq \left|\int_{t_0}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}g\left(s, x_n\big(\gamma(s)\big)\right)ds - \int_{t_0}^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}g\left(s, x_n\big(\gamma(s)\big)\right)ds\right|$$
$$+\left|\int_{t_0}^{t_1}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}g\left(s, x_n\big(\gamma(s)\big)\right)ds - \int_{t_0}^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}g\left(s, x_n\big(\gamma(s)\big)\right)ds\right|$$
$$= \left|\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}g\left(s, x_n\big(\gamma(s)\big)\right)ds\right|$$
$$+\left|\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}]g\left(s, x_n\big(\gamma(s)\big)\right)ds\right|$$
$$\leq \frac{\mathbb{M}_g}{\Gamma(\alpha)}\int_{t_1}^{t_2}|(t_2 - s)^{\alpha-1}|\,ds + \frac{\mathbb{M}_g}{\Gamma(\alpha)}\int_{t_1}^{t_2}|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}|\,ds$$
$$= \frac{\mathbb{M}_g}{\Gamma(\alpha)}a^{\alpha-1}(t_2 - t_1) + \frac{\mathbb{M}_g}{\Gamma(\alpha)}\int_{t_0}^{t_1}|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}|\,ds$$
$$\to 0$$

as $t_2 - t_1 \to 0$ uniformly for all $n \in \mathbb{N}$, where we use the dominated convergence theorem for the limit in the second term above. this implies that $\mathbb{B}x_n \to \mathbb{B}x$ uniformly Therefore $\mathbb{B}$ is partially continuous on $\mathbb{E}$.

Step IV. Next, we need to prove the remaining condition of the operator $\mathbb{B}$ in theorem 2.1 that is $\mathbb{B}$ is partially compact. Let $\mathbb{C}$ be the chain in $\mathbb{E}$.we shall prove that $\mathbb{B}(\mathbb{C})$ is uniformly bounded and equicontinuous in $\mathbb{E}$,

Let $y \in \mathbb{B}(\mathbb{C})$ be any arbitrary then $y = \mathbb{B}(x) for\ some\ x \in \mathbb{C}$. By hypothesis (H_6) ,we see that

$$|y(t)| = |(\mathbb{B}x)(t)| = \left|x_0 - f(t_0, x_0) + \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_n\big(\gamma(s)\big)\right) ds\right|$$

$$\leq |x_0 - f(t_0, x_0)| + \frac{\mathbb{M}_g}{\Gamma(\alpha)} \int_{t_0}^{t_1} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |(t_1 - s)^{\alpha-1}|\, ds$$

$$\leq |x_0 - f(t_0, x_0)| + \frac{\mathbb{M}_g}{\Gamma(\alpha)} (t_1 - t_0)$$

$$\leq |x_0 - f(t_0, x_0)| + \frac{\mathbb{M}_g}{\Gamma(\alpha)} a^{\alpha}$$

$$\leq: \mathbb{K}$$

for all $t \in \mathfrak{J}$.Hence we obtain $\|y(t)\| = \|\mathbb{B}(x)\| \leq \mathbb{K}$ for all $y \in \mathbb{B}(\mathbb{C})$is uniformly bounded. we next show that $\mathbb{B}(\mathbb{C})$ is equicontinuous. Let $y \in \mathbb{B}(\mathbb{C})$ be arbitrary and take $t_1, t_2 \in \mathfrak{J}$ with $t_1 \leq t_2$.we have

$$|(\mathbb{B}x_2)(t_2) - (\mathbb{B}x_1)(t_1)|$$

$$= \left|\int_{t_0}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_n\big(\gamma(s)\big)\right) ds - \int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_n\big(\gamma(s)\big)\right) ds\right|$$

$$\leq \left|\int_{t_0}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_n\big(\gamma(s)\big)\right) ds - \int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_n\big(\gamma(s)\big)\right) ds\right|$$

$$+ \left|\int_{t_0}^{t_1} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_n\big(\gamma(s)\big)\right) ds - \int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_n\big(\gamma(s)\big)\right) ds\right|$$

$$= \left|\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_n\big(\gamma(s)\big)\right) ds\right|$$

$$+ \left|\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] g\left(s, x_n\big(\gamma(s)\big)\right) ds\right|$$

$$\leq \frac{\mathbb{M}_g}{\Gamma(\alpha)} \int_{t_1}^{t_2} |(t_2 - s)^{\alpha-1}|\, ds + \frac{\mathbb{M}_g}{\Gamma(\alpha)} \int_{t_1}^{t_2} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}|\, ds$$

$$= \frac{\mathbb{M}_g}{\Gamma(\alpha)} a^{\alpha-1}(t_2 - t_1) + \frac{\mathbb{M}_g}{\Gamma(\alpha)} \int_{t_0}^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}|\, ds$$

$$\to 0$$

as $t_2 - t_1 \to 0$ uniformly $y \in \mathbb{B}(\mathbb{C})$. This means $\mathbb{B}(\mathbb{C})$ is equicontinuous. It follows that $\mathbb{B}(\mathbb{C})$ is relatively compact. Hence $\mathbb{B}$ is partially compact.

Step V: By hypothesis (H_7),by the fractional hybrid functional equation (1.1) has a lower solution of u is defined on $\mathfrak{J}$, such that ,

$$\frac{d^{\alpha}}{dt^{\alpha}}[u(t) - f(u(\alpha(t))] = g\big(t, u\big(\gamma(t)\big), t \in \mathfrak{J}$$

$$u(t_0) \leq x_0$$

By formulating a mild solution, we see that,

$$u(t) \leq f\left(t, u\big(\alpha(t)\big)\right) + x_0 - f(t_0, x_0) + \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x\big(\gamma(s)\big)\right) ds \qquad (4.1)$$

for $t \in \mathfrak{I}$. It follows that u satisfies the operator $u \leq \mathbb{A}u + \mathbb{B}u$.

Thus, we conclude that the operators $\mathbb{A}$ and $\mathbb{B}$ are satisfies all conditions of Theorem 2.1. Then the operator equation $\mathbb{A}x + \mathbb{B}x = x$ has a solution. Moreover, we have approximation of solution x_n as $n = 1,2,3, \ldots \ldots \ldots$ for the equation (1.1).

5.Conculsion

In this paper for proving the existence results however unlike existence the results, for nonlinear functional hybrid differential and integral equation discussed in Dhage[4,6,8].we gave the definitions of mild solution of the fractional order by using the Caputo fractional derivatives of order $\alpha \in (0,1)$ and then we discussed the existence of at least one mils solution of lower and upper solutions such that $u \leq v$ then the under condition theorem 2.1 it has corresponding solution $x_* and\ x^*$ ae these satisfies $x_* \leq \ x^*$ Hence they are the minimal and maximal solution of the hybrid functional $\mathbb{D}$.

6.Acknowledgment

The author is thankful to the refer and giving some suggestion a for the improvement of this paper and research work.

7.References

1. B. C. Dhage, “Partially considering mapping on partially ordered normed linear space and applications to functional integral equation”, Tamkang J. Math. 45(2) (2014) pp 397-420.
2. B. C. Dhage, “Quadratic perturbation of boundary value problem of second order ordinary differential equation”, Differ.Equat.Appl.2(4) (2010) pp 465-486.
3. B. C. Dhage, N. S. Jadhav, “Basic results on hybrid differential equations with linear perturbation of second type”, Tamkang J. Math 44(2) (2013) pp 171-186.
4. B. C. Dhage, Namdev S Jadhav, J. N. Salunke, Anurudra Y Shete, “Dhage Iteration Method for Nonlinear First order Hybrid Functional Differential equation of Second Type Linear Perturbation”, International journal of Mathematics and its Applications Vol.5.Issue4-E (2017) pp 605-614.
5. B.C. Dhage S.B. Dhage S.K. Ntoutys, “Approximating solutions of nonlinear hybrid differential equation”, Appl. Math Lett.,34 (2014) pp 76-80.
6. B.C. Dhage, “Operator theoretic techniques in the theory of nonlinear hybrid differential equations”, Linear Anal. forum 20 (2015) pp 15-31.
7. B.C. Dhage. “Hybrid fixed point theory in partially ordered normed linear space and applications to functional integral equation”, Tamkang J. Math. 44(2) (2012) pp 171-186.
8. B.C.Dhage, “ nonlinear D-set contraction mappings in partially ordered normed linear spaces and applications to functional hybrid integral equations”,Malaya Journal of Mathematik 3(1) (2015) pp 62-85.
9. Bai. Z Lu H, “positive solutions for boundary value problem of nonlinear fractional differential equation”, Journal Math anal Appl. 311(2)920050 PP 495-505.
10. Dussadee Somjai wang and Parinya Sa Ngiamsunthorn, “Existence and approximation of solutions to fractional order hybrid differential equations”, Advance if Difference equations (2016) 2016:278 PP 1-11.
11. H. Lu. S. Sun D Yang and H Tang, “Theory of fractional hybrid differential equation with linear perturbations of second type boundary value problem”, Computers and Mathematics with application2013 (2013) art.23-2013.

12. Herzallah, M.A.E. Baleau D, “On fractional order hybrid differential equation”, Abstr, Appl. Anal 2014 (2014) Art.id 389386 pp1-8.
13. Herzallah, M.A.E. El-Sayed and Baleau D, “Existence of a parodic mild solution for a nonlinear Neumann boundary value problem”, journal of computational analysis and application 15(2)(2013)pp341-352.
14. I. Podlubny, “Fractional Differential equation”, Academic press New York (1999).
15. Kilbas AA, Srivastava, HM, Trujillo, JJ, “Theory and application of fractional Differential equations”, Elsevier Science Inc New York (1974).
16. Oldham, KB, Spanier, J, “The fractional Calculus: Theory and application of Fractional Differentiation and Integration to arbitrary order Mathematics in Science and Engineering”, academic press, New York (1974).
17. S.G. Samko, A.A. Kilbas O.I. Marichev, “fractional integrals and Derivatives, Theory and application”, Gordon and Breach, New York (1993).
18. Sun S Zhao Y Han Z Li Y, “The Existence of solution for boundary value problem for fractional hybrid differential equation”, Commun. Nonlinear Sci number Simul.17(2) (2012) pp 4961-4967.
19. Zhao Y Sun S Han Z Li Q, “The existence of multiple positive solution for boundary value problem of nonlinear fractional differential equations”, Commun. Nonlinear Sci Numer Simmul.16(4) (2011) pp 2086-2097.
20. Zhao Y Sun,S Han Z,Li Q,“Theory of fractional hybrid differential equations”, Comput.Math.Appl.62(3) (2011)pp1312-1324.
21. Guo And Lakshmikanthan V, “Nonlinear Problems in Abstract cones”, Academic press New York (1998).
22. Heikkila S. Lakshmikanthan V, “Monotone iterative technique for discontinuous nonlinear differential equations ‘, Mark Dekker Inc., New York (1994).