Solution of Burger's Equation Arising in the Longitudinal Dispersion Phenomenon in Fluid Flow through Porous Media Using Elzaki Transform Method

Krima B. Patel[1] and Dr. Kamlesh A. Patel[2*]

[1]Research Scholar, Mathematics Department, Shri U. P. Arts, Smt. M. G. Panchal Science & Shri V. L. Shah Commerce college, Pilvai, India

[2]Associate Professor, Mathematics Department, Shri U. P. Arts, Smt. M. G. Panchal Science & Shri V. L. Shah Commerce college, Pilvai, India

Abstract: *In this paper we have discussed about numerical solution of non-linear Burger's equation for longitudinal dispersion phenomenon in miscible fluid flow through porous media. We have used Elzaki Decomposition Method for solving Burger's equation. This method is basically combination of two methods Elzaki transform and Adomian Decomposition method. We can easily handle non-linear terms arising in Burger's equation with the help of Decomposition method. The purpose of this study is to show the applicability and efficiency of this mixture method. The graphical illustration of the dispersion problem is presented in paper.*

Keywords: **Elzaki transform, Adomian Decomposition Method, Burger's equation**

1. Introduction

The non-linear equations are the most important phenomena across the world. Non-linear phenomena have important efficiency on applied mathematics, physics and issues related to engineering. Still there is a big problem of obtaining the exact solution of non-linear partial differential equations in physics and applied mathematics. Due to this reason many researchers focus to develop new methods. The longitudinal dispersion phenomenon is the process by which miscible fluid flow disperse in the direction of flow. The problem of miscible fluid flow can be seen in coastal areas, where the ground waterbeds are gradually displaced by seawater. Longitudinal dispersion phenomenon plays an important role to control salinity of the soil in coastal areas. Many researchers have discussed the longitudinal dispersion phenomenon with different point of views like as Tarig M. Elzai and Hwajoon Kim [4] used EHPM to solve Burger's equation. Ravi Borana, Vikas Pradhan and Manoj Mehta [5] proposed Crank-Nicolson scheme. Kunjan Shah and Twinkle Singh [6] applied new integral transform and Homotopy Perturbation method. M. O. Olayiwola [7] presented MVIM for solution of non-linear Burger's equation arising in longitudinal dispersion phenomena in fluid flow through porous media. Ramakanta Meher, M.N. Mehta and S.K. Meher [8,9,10] solved the dispersion problem by means of a new approach to Backlund transformation of non-linear partial differential equation. Also, they used Adomian decomposition method in

* Corresponding Author

other work. Mahendra A. Patel and N.B. Desai [11] discussed Burger's equation for longitudinal dispersion phenomenon using Homotopy analysis method.

2. Preliminaries

2.1 Elzaki Transform

A New integral transform called Elzaki transform defined for functions of exponential order is proclaimed. We consider functions in the set A defined by

$$A = \left\{ f(t) : \exists M, K_1, K_2 > 0, |f(t)| < Me^{\frac{|t|}{K_j}}, if\ t \in (-1)^j \times [0, \infty) \right\}$$

Definition: If $f(t)$ is function defined for all $t \geq 0$, its Elzaki transform is the integral of $f(t)$ times $e^{-\frac{t}{v}}$ from $t = 0$ to ∞. It is a function of v and is defined as

$$E[f(t)] = T(v) = v \int_0^\infty f(t) e^{-\frac{t}{v}}\, dt \quad , \quad v \in (K_1, K_2)$$

Or equivalently

$$T(v) = v^2 \int_0^\infty f(vt) e^{-t}\, dt \ , \quad K_1, K_2 > 0$$

Properties of Elzaki transform

1. $E(1) = v^2$
2. $E(t^n) = n!\, v^{n+2}$
3. $E(t) = v^3$
4. $E^{-1}(v^{n+2}) = \frac{t^n}{n!}$
5. $E\left[u^{(m)}(t)\right] = \frac{1}{v^m} T(x, v) - \sum_{k=0}^{m-1} v^{2-m-k}\, u^{(k)}(0)\ ,\ m \geq 1$

2.2. Elzaki Decomposition Method

Consider a general nonlinear differential equation

$$\mathrm{L}S_w(x,\mathrm{T}) = \mathrm{N}S_w(x,\mathrm{T}) + \mathrm{g}(x,\mathrm{T}) \tag{i}$$

Where, L is the 1st order linear differential operator. N is the nonlinear differential operator, S_w is the dependent variable, x and T are independent variables, g(x,T) is the source term.

Apply Elzaki Transform to the equation (i)

$$\mathrm{E}(\mathrm{L}S_w(x,\mathrm{T})) = \mathrm{E}(\mathrm{N}S_w(x,\mathrm{T})) + \mathrm{E}(\mathrm{g}(x,\mathrm{T}))$$

Using differentiation property of the Elzaki Transform, we have

$$\frac{\mathrm{E}(S_w(x,\mathrm{T}))}{v} - vS_{w0}(x,\mathrm{T}) = \mathrm{E}(\mathrm{N}S_w(x,\mathrm{T})) + \mathrm{E}(\mathrm{g}(x,\mathrm{T}))$$

$$\mathrm{E}(S_w(x,\mathrm{T})) = v^2 S_{w0}(x,\mathrm{T}) + vE(\mathrm{N}S_w(x,\mathrm{T})) + v\mathrm{E}(\mathrm{g}(x,\mathrm{T})) \tag{ii}$$

Using inverse Elzaki Transform on both side of the equation (ii), we get

$$S_w(x, \mathrm{T}) = \mathrm{G}(x, \mathrm{T}) + E^{-1}[v\mathrm{E}(\mathrm{N}S_w(x, \mathrm{T}))] \quad \text{(iii)}$$

Where, G(x,T) represents the term arising from the source term and the prescribed initial conditions.

The representation of the solution (iii) as an infinite series is given below,

$$S_w(x,\mathrm{T}) = \sum_{N=0}^{\infty} S_{wn}(x, \mathrm{T}) \quad \text{(iv)}$$

The non-linear term has been decomposed as:

$$\mathrm{N}\, S_w(x,\mathrm{T}) = \mathrm{A}_n (S_{w0}, S_{w1}\, S_{w2}, \ldots.\, S_{wn}) \quad \text{(v)}$$

Where, A_n are the Adomian polynomials of functions S_{w0}, S_{w1} S_{w2}, …. S_{wn} and can be calculated by formula given as:

$$\mathrm{A}_n = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\left[\sum_{k=0}^{\infty} \lambda^k S_{wk}\right]_{\lambda=0} \qquad n = 0, 1, 2, \ldots\ldots$$

Substituting (iv) and (v) into (iii),

$$S_{wn}(x,\mathrm{T}) = \mathrm{G}(x,\mathrm{T}) + \mathrm{E}^{-1}[v\, \mathrm{E}(\mathrm{A}_n)]$$

Where, $S_{w0}(x,\mathrm{T}) = \mathrm{G}(x,\mathrm{T})$

$$S_{wn+1}(x,\mathrm{T}) = \mathrm{E}^{-1}[v\, \mathrm{E}(\mathrm{A}_n)] \qquad n = 0, 1, 2, \ldots\ldots \quad \text{(vi)}$$

Where, the Elzaki Transform and the inverse Elzaki Transform are applied on (vi) respectively, the iterations S_{w0}, S_{w1} S_{w2}, …. S_{wn} were obtained, which in turn gave the general solution as,

$$S_w(x,\mathrm{T}) = S_{w0}(x,\mathrm{T}) + S_{w1}(x,\mathrm{T}) + S_{w2}(x,\mathrm{T}) + \ldots\ldots\ldots\ldots$$

3. Burger's equation

3.1 Statement of the problem

Miscible displacement in porous media is a type of double-phase flow in which two phases are completely soluble in each other. Therefore, capillary forces between the fluids do not come onto effect. The longitudinal dispersion of the contaminated or saline water with the concentration C (x, t) flowing in the x-direction has been considered, the homogeneous porous medium is saturated with fresh water. The miscible flow (contaminated or saline and fresh water) under conditions of complete miscibility could be though to behave, locally at least, as a single-phase fluid, which would obey Darcy's law. The change of concentration, in turn, would be caused by diffusion along the flow channels and thus be governed by the bulk coefficients of diffusion of the one fluid in the other. There is no mass transfer between the solid and liquid phases. The miscible flow takes place both longitudinally and transversely, but the spreading caused by dispersion is greater in the direction of flow than the transverse direction.

3.2 Mathematical Formulation of the Problem

According to Darcy's law, the equation of continuity for the mixture is,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{v}) = 0$$

Where, ρ is the density for the mixture and $\bar{v}$ is the pore seepage velocity.

The equation of diffusion for a fluid flow through a homogeneous porous medium, without increasing or decreasing the dispersing material is given by

$$\frac{\partial C}{\partial t} + \nabla \cdot (C\bar{v}) = \nabla \cdot \left[\rho \bar{D} \nabla \left(\frac{C}{\rho}\right)\right] \tag{1}$$

Where C is the concentration of the fluids, $\bar{D}$ is the tensor coefficients of dispersion with nine components D_{ij}

In a laminar flow through homogeneous porous medium at a constant temperature, ρ is constant.
Then $\nabla \cdot \bar{v} = 0$

Therefore equation (1) becomes,

$$\frac{\partial C}{\partial t} + \bar{v} \cdot \nabla C = \nabla \cdot (\bar{D} \nabla C) \tag{2}$$

When the seepage velocity is the along x-axis, the nonzero components are $D_{11} = D_L \cong \gamma$ (coefficients of longitudinal dispersion, is a function of x along the x-axis) and other D_{ij} are zero

In this case, equation (2) becomes,

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = \gamma \frac{\partial^2 C}{\partial x^2} \tag{3}$$

Where u is the component of velocity along the x- axis, which is time dependent as well as concentration along the x-axis in x≥0 direction and $D_L \geq 0$ and it is cross sectional flow velocity of porous medium. Therefore,

$$u = \frac{C(x,t)}{C_0}$$

Where $x > 0$ and for $C_0 \cong 1$

Hence equation (3) becomes

$$\frac{\partial C}{\partial t} + C \frac{\partial C}{\partial x} = \gamma \frac{\partial^2 C}{\partial x^2}$$

This is the non-linear Burger's equation for longitudinal dispersion of miscible fluid flow through porous media.

3.3 Solution of the Problem

Our problem is,

$$\frac{\partial C}{\partial t} + C \frac{\partial C}{\partial x} = \gamma \frac{\partial^2 C}{\partial x^2}$$

$$\therefore C_t = \gamma C_{xx} - C C_x \tag{1}$$

Taking initial condition $C(x,t) = e^{-x}$

Applying Elzaki transform (1)

$E(C) = v^2 C_0 + vE[\gamma C_{xx} - CC_x]$

Now, applying inverse Elzaki transform we get,

$C = C_0 + E^{-1}v[E(\gamma C_{xx} - CC_x)]$

$\therefore C = C_0 + E^{-1}v[E(\emptyset_1 - \emptyset_2)]$

Where $\emptyset_1 = \sum_{n=0}^{\infty} A_n$ and $\emptyset_2 = \sum_{n=0}^{\infty} B_n$

$$\therefore C = C_0 + E^{-1}v\left[E\left(\sum_{n=0}^{\infty} A_n - \sum_{n=0}^{\infty} B_n\right)\right]$$

$$\therefore C = C_0 + E^{-1}v[E(\textstyle\sum_{n=0}^{\infty}(A_n - B_n))] \qquad (2)$$

The Adomian polynomials A_n and B_n are calculated as

$A_n = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\left[\emptyset_1\left(\sum_{k=0}^{\infty}\lambda^k C_k\right)\right]_{\lambda=0}$ $\qquad$ n = 0, 1, 2,…….

$B_n = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\left[\emptyset_2\left(\sum_{k=0}^{\infty}\lambda^k C_k\right)\right]_{\lambda=0}$ $\qquad$ n = 0, 1, 2,…….

Taking $\gamma = 1$ the first three components of these polynomials are,

$A_0 = (C_0)_{xx}$

$A_1 = (C_1)_{xx}$

$A_2 = (C_2)_{xx}$

$B_0 = C_0(C_0)_x$

$B_1 = C_0(C_1)_x + C_1(C_0)_x$

$B_2 = C_0(C_2)_x + C_1(C_1)_x + C_2(C_0)_x$

Other polynomials can be generated in like this manner, substituting this decomposition series in (2) and comparing both sides coefficients, we get,

$C_0 = e^{-x}$

$C_1 = E^{-1}v[E(A_0 - B_0)]$

$\quad = (e^{-2x} + e^{-x})T$

$C_2 = E^{-1}v[E(A_1 - B_1)]$

$\quad = (3e^{-3x} + 6e^{-2x} + e^{-x})\frac{T^2}{2!}$

$C_3 = E^{-1}v[E(A_2 - B_2)]$

$$= (16e^{-4x} + 51e^{-3x} + 28e^{-2x} + e^{-x})\frac{T^3}{3!}$$

Finally, the solution is given as,

$$C = \sum_{n=0}^{\infty} C_n$$

$$= e^{-x} + (e^{-2x} + e^{-x})T + (3e^{-3x} + 6e^{-2x} + e^{-x})\frac{T^2}{2!} + (16e^{-4x} + 51e^{-3x} + 28e^{-2x} + e^{-x})\frac{T^3}{3!} + \cdots$$

4. Numerical solution

The following table shows the approximate concentration of contaminated or saline water for different values of position at different time using Elzaki decomposition method.

Table 1. Concentration of contaminated or saline water

X	t=0	t=0.1	t=0.2	t=0.3	t=0.4	t=0.5	t=0.6	t=0.7	t=0.8	t=0.9	t=1
0	1	1	1	1	1	1	1	1	1	1	1
0.1	0.8494	0.8393	0.8284	0.819	0.8116	0.8059	0.8014	0.7951	0.7951	0.7928	0.7909
0.2	0.7132	0.6971	0.6799	0.6652	0.6536	0.6446	0.6376	0.6321	0.6277	0.6242	0.6212
0.3	0.59	0.571	0.5509	0.5338	0.5231	0.5099	0.5018	0.4955	0.4904	0.4863	0.4828
0.4	0.4784	0.459	0.4385	0.4212	0.4075	0.397	0.3888	0.3823	0.3772	0.373	0.3696
0.5	0.3775	0.3593	0.3403	0.3241	0.3115	0.3017	0.2941	0.2882	0.2834	0.2796	0.2764
0.6	0.2862	0.2705	0.2541	0.2402	0.2294	0.221	0.2145	0.2094	0.2054	0.2021	0.1994
0.7	0.2036	0.1912	0.1783	0.1674	0.1589	0.1524	0.1473	0.1433	0.1402	0.1376	0.1355
0.8	0.1288	0.1203	0.1114	0.104	0.0982	0.0937	0.0902	0.0875	0.0854	0.0836	0.0822
0.9	0.0612	0.0568	0.0523	0.0486	0.0456	0.0434	0.0416	0.0402	0.0392	0.0383	0.0375
1	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001

5. Graphical Representation

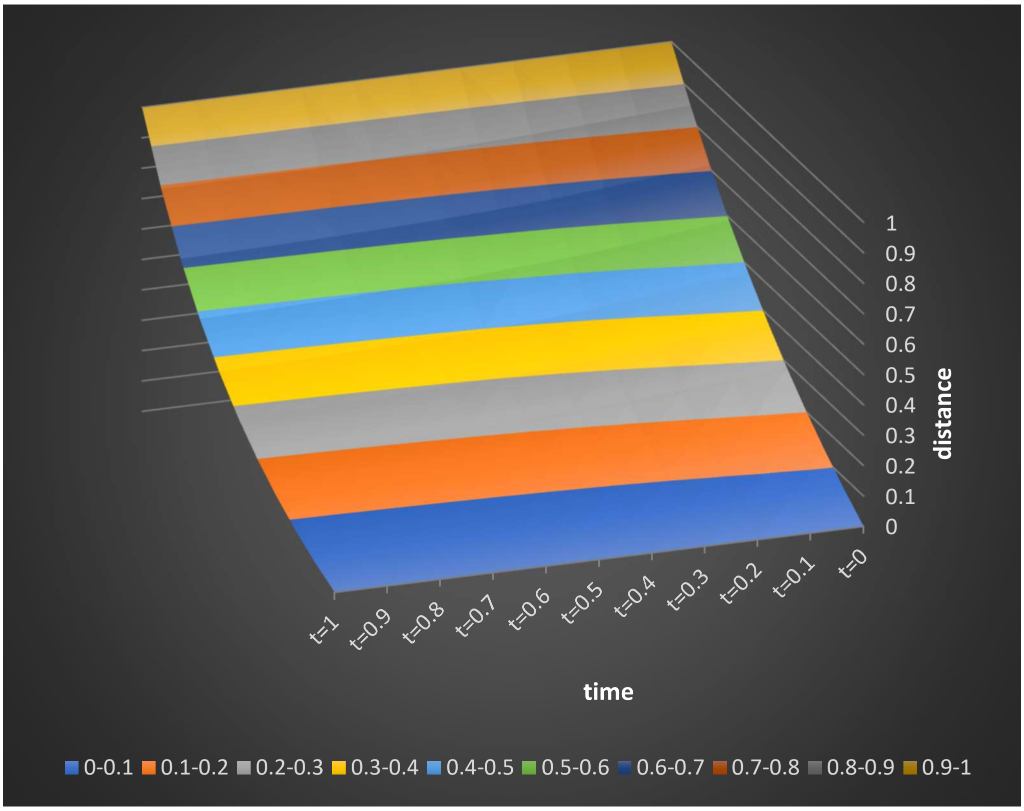

Figure 1. Concentration of contaminated or saline water

6. Conclusion

The graphical and numerical solutions have been obtained to predict the possible concentration of the contaminated or saline water in unsteady unidirectional seepage flow through homogeneous isotropic porous media. Clearly, the graph indicates that as distance x and time t increases the concentration of the contaminated water gradually decreases. The concentration of the contaminated water decreases as the distance x increases for the given time $t>0$. Here the initial concentration of contaminated water at $x=0$ is highest and it decreases as distance x increases for given time $t>0$.

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