A note on basic results on relative ordered Γ-ideals in ordered LA-Γ-semigroups

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ABSTRACT. In this paper, we introduce the concept of relative ordered Γ-ideals, relative ordered quasi-Γ-ideals, relative ordered bi-Γ-ideals in ordered LA-Γ-semigroups. We characterize ordered LA-Γ-semigroups, LA*-Γ-semigroups, ordered LA-Γ-groups and left (resp. right) relative simple ordered LA-Γ-semigroups by the relative ordered Γ-ideals in ordered LA-Γ-semigroups.

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1 Prerequisites

A non-associative and non-commutative algebraic system called left almost semigroup, also known as an LA-semigroup, is an algebraic structure, in midway between a commutative semigroup and a groupoid. This algebraic system was introduced by M. A. Kazim and M. Naseeruddin [11] in 1972. This structure is also known as Abel-Grassmann's groupoid (AG-groupoid) [24], and as an invertive groupoid [20]. Mushtaq and Yusuf [21] studied the concept of ideals in LA-semigroups. In 2015, Yahya and Basar [12] studied ideals in LA-Γ-semigroups. In 2019, Basar [7] studied (m, n)-Γ-ideals in ordered LA-Γ-semigroups. In 2020, Kausar et al [15] studied ordered LA-groups and ideals in ordered LA-semigroups and developed these nice ideal theoretic results in ordered LA-groups as well as in ordered LA-semigroups .

A groupoid G is called medial if $(ab)(cd) = (ac)(bd)$ [10]. It is called paramedial if (ab)(cd) = (db)(ca) [22]. Moreover, an LA-semigroup is medial but, in general, an LA-semigroup may not be paramedial [11]. An LA-semigroup with left identity is paramedial and it further satisfies the following [24]:

$$a(bc) = b(ac), \text{ and } (\text{ab})(\text{cd}) = (\text{dc})(\text{ba}).$$

As a generalization of semigroups and ternary semigroups, the concept of Γ-semigroup was introduced by M. K. Sen [13]. A non-empty subset S of an ordered LA-Γ-semigroup G is an LA-Γ-subsemigroup of G if $S^2 = S\Gamma S \subseteq S$.

The concept of relative ideals was given by Wallace [8] and [9]. We recall that if S is a semigroup, $\varphi \neq B \subseteq S$. Then a left B-ideal (relative left ideal) of S is a nonempty set $A \subseteq S$ such that $BA \subseteq A$. Similarly, one can define a right B-ideal(relative right ideal) and a two-sided B-ideal (relative ideal) of S.

The concept of quasi-ideals was given by Steinfeld [16], [17], [18] in rings and in semigroups. The concept of bi-ideal was introduced by Good and Hughes [19]. It is well known that the concept of a one-sided ideal of rings and semigroups is a generalization of the concept of an ideal of rings and semigroups and the concept of a quasi-ideal of semigroups and rings is a generalization of a one sided ideal of semigroups and rings, and bi-ideals are generalization of quasi-ideals.

N. M. Khan and M. F. Ali [14] introduced relative ideals in ordered semigroups and deduced nice results. Thereafter, relative ideals has been introduced and studied in different algebraic structures by many mathematicians.

In 2020, Basar [4] introduced and studied relative hyperideals, relative bi-hyperideals, relative quasi-hyperideals, relative prime hyperideals, relative weakly prime hyperideals, relative semiprime hyperideals, relative prime and relative semiprime bi-hyperideals, and hyper relative regularity.

Dixit and Diwan [23] studied quasi and bi-ideals in ternary semigroups. Basar, Yaqoob, Yahya and Sabahat Ali Khan [1] studied ordered involution Γ-semihypergroups by weakly prime Γ-hyperideals. In 2021, Basar, Yaqoob, Satyanarayana and Yahya [5] introduced the relative left, right, lateral, two-sided hyperideal, relative quasi-hyperideal, relative bi-hyperideal, relative sub-idempotent ordered bi-hyperideal, relative generalized quasi-hyperideal, relative generalized bi-hyperideal, relative regularity of ordered ternary semihypergroups and relative left (right, lateral) simple ordered ternary semihypergroups. Then, in the same year, Basar, Satyanarayana and Kumar [6] introduced relative act hyperideals in ordered semihypergroups.

Furthermore, in 2021, Basar, Satyanarayana, Kuncham, Kumar and Yahya [2] introduced relative ideals in abstract affine Γ-near rings. Moreover, relative quasi-Γ-ideals and relative bi-Γ-ideals in Γ-near rings are introduced and studied in [3] by Basar, Yahya, Satyanarayana and Talee.

We now introduce the following concepts in ordered LA-Γ-semigroups. An ordered LA-Γ-smigroup is denoted by G unless otherwise specified.

Definition 1.1. Suppose that G is an ordered LA-Γ-semigroup and $S \subseteq G$. Then a non-empty subset I of G is a relative ordered left (resp. right) Γ-ideal of G if

(1) $S\Gamma I \subseteq I$ (resp. $I\Gamma S \subseteq I$), and

(2) If $a \in I$ and $b \in S$ such that $b \leq a \Rightarrow b \in I$, or equivalently, $(I]_S \subseteq I$.

Furthermore, I is called a relative ordered Γ-ideal of G if I is both a left and a right relative ordred Γ-ideal of G. Also, a relative ordered Γ-ideal (left, right, two-sided) is an LA-Γ-subsemigroup but not vice versa in general.

Definition 1.2. A non-empty subset I of G is a relative ordered interior Γ-ideal of G if $(S\Gamma I)\Gamma S \subseteq I$ and $(I]_S \subseteq I$.

Definition 1.3. A non-empty subset Q of G is a relative ordered quasi-Γ-ideal of G if $(Q\Gamma S]_S \cap (S\Gamma Q]_S \subseteq Q$ and $(Q]_S \subseteq Q$, where $S \subseteq G$.

Definition 1.4. An LA-Γ-subsemigroup B of G is a relative ordered bi-Γ-ideal of G if $(B\Gamma S)\Gamma B \subseteq B$ and $(B]_S \subseteq B$, where $S \subseteq G$.

Definition 1.5. A non-empty subset B of G is a relative ordered generalized bi-Γ-ideal of G if $(B\Gamma S)\Gamma B \subseteq B$ and $(B]_S \subseteq B$, where $S \subseteq G$.

Definition 1.6. A groupoid $(G, \cdot)$ is a left almost Γ-group, i.e., LA-Γ-group if

(1) $(a \cdot \alpha \cdot \beta \cdot b) \cdot \gamma \cdot c = (c \cdot \alpha \cdot \beta \cdot b) \cdot \gamma \cdot a$ for all $a, b, c \in G$;

(2) There exists $e \in G$ such that $e \cdot \gamma \cdot a = a$ for every $a \in G$;

(3) For every $a \in G$ there exists $a^{'} \in G$ such that $a^{'} \cdot \alpha \cdot \beta \cdot a = e$,

for $\alpha, \beta, \gamma \in \Gamma$.

It is clearly observable that the left identity 'e' and the left inverse are unique. We also see that the left inverse is also right inverse since $a\alpha a^{'} = (e\beta a)\gamma a^{'} = (a^{'}\alpha a)\beta e = e\gamma e = e$. It then follows that a$^{'}$ is a right inverse of a as well.

Definition 1.7. An ordered LA-Γ-group $(G, \Gamma, \cdot, \leq)$ is a poset, at the same time an LA-Γ-group such that $a \leq b \Rightarrow a\alpha c \leq b\beta c$ and $c\alpha a \leq c\beta b$ for all $a, b, c \in G$, and $\alpha, \beta, \gamma \in \Gamma$.

Suppose that G is an ordered LA-Γ-group(po-LA-Γ-group), then, clearly, if $a \leq b$, then $b^{-1} \leq a^{-1}$ for $a, b \in G$.

Definition 1.8. An LA-Γ-subsemigroup T of G is a relative ordered (1, 2)-Γ-ideal of G if $(T\Gamma S)\Gamma T^2 \subseteq T$ and $(T]_S \subseteq T$, where $S \subseteq G$.

2 Relative ordered Γ-ideals in ordered LA-Γ-semigroups

In this section, we make to develop relative ordered ideal theory in the non-associative and non-commutative ordered algebraic structure called LA-Γ-semigroup. In the following Lemma, we begin to prove as to when a relative ordered right Γ-ideal of an ordered LA-Γ-semigroup is a relative ordered Γ-ideal.

Lemma 2.1. *A relative ordered right* Γ*-ideal of an ordered LA-*Γ*-semigroup* G *with left identity* e *is a relative ordered* Γ*-ideal of* G*.*

Proof. Suppose that A is a relative ordered right Γ-ideal of G and $S \subseteq G$. It then follows that $(A]_S \subseteq A$. Suppose that $a \in A$ and $s \in S$. Therefore, we have the following:

$$s\alpha a = (e\beta s)\gamma a = (a\alpha s)\beta e \in A\Gamma S \in A$$

for $\alpha, \beta, \gamma \in \Gamma$. Hence A is a relative ordered Γ-ideal of G. □

In the next Lemma, we prove the relationship between a relative ordered Γ-ideal of an ordered LA-Γ-semigroup and a relative ordered interior Γ-ideal of an ordered LA-Γ-semigroup.

Lemma 2.2. *Any relative ordered* Γ*-ideal of an ordered LA-*Γ*-semigroup* G *is a relative ordered interior* Γ*-ideal of* G*.*

Proof. Suppose that A is a relative two sided ordered Γ-ideal of G, and $S \subseteq G$. Then, we have $(A]_S \subseteq A$. Thus $(S\Gamma A)S \subseteq A\Gamma S \subseteq A$. Hence A is a relative ordered interior Γ-ideal of G. □

In the following Theorem, we prove the necessary and sufficient condition for a non-empty subset of an ordered LA-Γ-semigroup to be a relative ordered interior Γ-ideal of an ordered LA-Γ-semigroup.

Theorem 2.3. *Suppose that* G *is an ordered LA-*Γ*-semigroup with left identity* e*. Then a non-empty subset* A *of* G *is a relative ordered interior* Γ*-ideal of* G *if* A *is a relative ordered* Γ*-ideal of* G *and conversely.*

Proof. ⇒. Suppose that A is a relative ordered interior Γ-ideal of G and $S \subseteq G$. Then we receive $(A]_S \subseteq A$. Suppose that $a \in A$ and $s \in S$. Therefore, we further receive the following:

$$a\alpha s = (e\beta a)\gamma s \in (S\Gamma A)\Gamma S \subseteq A \text{ for } \alpha, \beta, \gamma \in \Gamma.$$

So, A is a relative ordered right Γ-ideal of G. Hence A is a relative ordered Γ-ideal of G by Lemma 2.1.
⇐. It follows by Lemma 2.2. □

Following Lemma gives us the connection between one sided or both sided relative ordered Γ-ideals, and relative ordered bi-Γ-ideal in the setting of ordered LA-Γ-semigroups.

Lemma 2.4. *Any right, or left, or both sided, relative ordered* Γ*-ideal of* G *is a relative ordered bi-*Γ*-ideal of* G*.*

Proof. Suppose that R is a relative ordered right Γ-ideal of G and $S \subseteq G$. It then follows that $(R]_S \subseteq R$. Therefore, $(R\Gamma S)\Gamma R \subseteq R\Gamma S \subseteq R$. Hence R is a relative ordered bi-Γ-ideal of G.
Similarly, it can be proved for relative ordered left Γ-ideal and relative ordered Γ-ideal. □

In the following Lemma, we prove the relative relationship between a left or a right, or a two-sided relative ordered Γ-ideal and a relative ordered quasi-Γ-ideal.

Lemma 2.5. *Any left or right, or a two-sided relative ordered Γ-ideal of G is a relative ordered quasi-Γ-ideal of G.*

Proof. Suppose that R is a relative ordered right Γ-ideal of G, and $S \subseteq G$. It then follows that $(R]_S \subseteq R$. Therefore, we receive the following:

$$(R\Gamma S]_S \cap (S\Gamma R]_S \subseteq (R\Gamma S]_S \subseteq (R]_S \subseteq R.$$

Hence R is a relative ordered quasi-Γ-ideal of G. In a similar fashion, it can be proved for a relative ordered left Γ-ideal, and a relative ordered Γ-ideal. □

The following Proposition gives us information regarding the interplay between a relative ordered quasi-Γ-ideal, and an ordered LA-Γ-subsemigroup.

Proposition 2.6. *Any relative ordered quasi-Γ-ideal of G is an LA-Γ-subsemigroup of G.*

Proof. Suppose that Q is a relative ordered quasi-Γ-ideal of G, and $S \subseteq G$. It then follows that

$$Q\Gamma Q \subseteq Q\Gamma S \subseteq (Q]_S\Gamma(S]_S \subseteq (Q\Gamma S]_S \text{ and } \mathrm{Q\Gamma Q \subseteq S\Gamma Q \subseteq (S]_S\Gamma(Q]_S \subseteq (S\Gamma Q]_S}.$$

Thus, we have the following:

$$Q^2 = Q\Gamma Q \subseteq (Q\Gamma S]_S \cap (S\Gamma Q]_S \subseteq Q.$$

Hence Q is an ordered LA-Γ-subsemigroup of G. □

The following Theorem shows the intersection property of a relative ordered left Γ-ideal, and a relative ordered right Γ-ideal in terms of relative ordered quasi-Γ-ideal in the constructive algebraic home of ordered LA-Γ-semigroups.

Theorem 2.7. *Suppose that R and L, respectively, is a right and a left relative ordered Γ-ideal of an ordered LA-Γ-semigroup G. Then $R \cap L$ is a relative ordered quasi-Γ-ideal of G.*

Proof. Suppose that $S \subseteq G$. As R is a relative ordered right Γ-ideal and L is a relative ordered left Γ-ideal of G, we receive the following:

$$((R \cap L)\Gamma S]_S \cap (S\Gamma(R \cap L)]_S \subseteq (R\Gamma S]_S \cap (S\Gamma L]_S \subseteq (R]_S \cap (L]_S \subseteq R \cap L.$$

Furthermore, we have $(R \cap L]_S \subseteq R \cap L$. Hence $R \cap L$ is a relative ordered quasi-Γ-ideal of G. □

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The following Lemma makes it clear that no proper interior or bi-, or generalized bi-, or quasi- relative ordered Γ-ideal is contained an ordered LA-Γ-semigroup G, whenever G is an ordered LA-Γ-group.

Lemma 2.8. *Suppose that an ordered LA-Γ-semigroup $(G, \Gamma, \cdot, \leq)$ with left identity e is an ordered LA-Γ-group. Then no proper interior or bi-, or generalized bi-, or quasi-relative ordered Γ-ideal is contained in G.*

Proof. Suppose that I is a relative ordered interior Γ-ideal of G, and $S \subseteq G$. Further, suppose that $s \in S$ and $i \in I$. It then follows that there exists $i^{-1} \in S$ satisfying $i\alpha i^{-1} = i^{-1}\beta i = e$, where e is the left identity of G and $\alpha, \beta \in \Gamma$. Then, we have the following:

$$s = e\alpha s = (i^{-1}\beta i)\gamma s = (s\delta i)i^{-1} \in (S\Gamma I)\Gamma S \subseteq I,$$

for $\alpha, \beta, \gamma, \delta \in \Gamma$. Therefore, $S \subseteq I \Rightarrow S = I$.

Secondly, let B be a relative ordered bi-Γ-ideal of G and $S \subseteq G$. Assume $s \in S$ and $b \in B$. It then follows that there exists $b^{-1} \in S$ satisfying $b\alpha b^{-1} = b^{-1}\beta b = e$, where e is the left identity of G and $\alpha, \beta \in \Gamma$. So, we receive the following:

$$\begin{aligned} s = (e\gamma_1 e)\gamma_2 s &= (s\gamma_3 e)\gamma_4 e \\ &= (s\gamma_5(b\gamma_6 b^{-1}))\gamma_7(b\gamma_8 b^{-1}) \\ &= (b\gamma_9(s\gamma_{10} b^{-)})\gamma_{11}(b\gamma_{12} b = ((b\gamma_{13} b^{-1})\gamma_{14}(s\gamma_{15} b^{-1}))\gamma_{16} b \\ &= ((b\gamma_{17} s)\gamma_{18}(b^{-1}\gamma_{19} b^{-1}))\gamma_{20} b = ((b^{-1}\gamma_{21} s)\gamma_{22}(b^{-1}\gamma_{23} b))\gamma_{24} b \\ &= ((b^{-1}\gamma_{25} s)\gamma_{26}(b\gamma_{27} b^{-1}))\gamma_{28} b, \text{ since } \mathrm{b}^{-1}\alpha\mathrm{b} = \mathrm{b}\beta\mathrm{b}^{-1}, \\ &= b(\gamma_{29}((b^{-1}\gamma_{30} s)\gamma_{31} b^{-1}))\gamma_{32} b \in (B\Gamma S)\Gamma B \subseteq B, \end{aligned}$$

for $\alpha, \beta, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{15}, \gamma_{16}, \gamma_{17}, \gamma_{18}, \gamma_{19}, \gamma_{20}, \gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{24}, \gamma_{25}, \gamma_{26}, \gamma_{27}, \gamma_{28}, \gamma_{29}, \gamma_{30}, \gamma_{31}, \gamma_{32} \in \Gamma$. Therefore $S \subseteq B$. It follows that $S = B$.

Finally, assume that Q is a relative ordered quasi-Γ-ideal of G, $0 \neq q \in Q$ and $S \subseteq G$. Then we have the following:

$$e = qq^{-1} = q^{-1}q \in Q\Gamma S \cap S\Gamma Q \subseteq (Q\Gamma S]_S \cap (S\Gamma Q]_S \subseteq Q.$$

This implies that $e \in Q$. Now suppose that $s \in S$. Then we have the following:

$$s = e\alpha s = e\beta(e\gamma s) = (s\delta e)\theta e \in Q\Gamma S \; S\Gamma Q \subseteq (Q\Gamma S]_S \cap (S\Gamma Q]_S \subseteq Q,$$

for $\alpha, \beta, \gamma, \delta, \theta \in \Gamma$. It follows that $S \subseteq Q$. Hence $Q = S$. □

In the following Proposition, we show the necessary and sufficient condition for a right cancellative ordered LA-Γ-semigroup to be an ordered LA-Γ-group.

Proposition 2.9. *Suppose that G is a right cancellative ordered LA-Γ-semigroup with left identity e. Then G is an ordered LA-Γ-group if and only if $G\gamma g = G$ for all $g \in G$ and $\gamma \in \Gamma$.*

Proof. $\Rightarrow$ Suppose that G is an ordered LA-Γ-group, and $g \in G$. Then we have

$$g = e\alpha g = (b\beta b^{-1})\gamma g = (g\delta b^{-1})\theta b \in G\Gamma b,$$

for $\alpha, \beta, \gamma, \delta, \theta \in \Gamma$. This implies that $G \subseteq G\Gamma b$. Hence $G\gamma b = G$ for all $b \in G$, and $\gamma \in \Gamma$.

$\Leftarrow$ Suppose that G is a right cancellative ordered LA-Γ-semigroup such that $G\Gamma g = G$ for all $g \in G$. Then it follows from the given condition that $p\gamma g = e$ for some $p \in G$, and $\gamma \in \Gamma$. It follows that there exists left inverse of $g \in G$ given that g is an arbitrary member of G. For preserving the uniqueness, suppose that $q\gamma g = e$, where $q \in G$ and $\gamma \in \Gamma$. Furthermore, we have the following: $p\alpha g = q\alpha g \Rightarrow p = q$ by the right cancellative law for $\alpha \in \Gamma$. Hence G is an ordered LA-Γ-group. □

3 Characterization of Ordered LA*-Γ-semigroups and Simple Ordered LA-Γ-semigroups by Relative Ordered Γ-Ideals

In this section, we provide a new characterization of ordered LA*-Γ-semigroups and simple ordered LA-Γ-semigroups in terms of relative ordered interior Γ-ideals as well as relative ordered bi-Γ-ideals. We begin by proving the following Proposition which gives the characterizing property when an ordered LA*-Γ-semigroup is an ordered LA*-Γ-group.

Proposition 3.1. *Suppose that an ordered LA*-Γ-semigroup G with left identity e is an ordered LA*-Γ-group. Then $G\Gamma g = g\Gamma G = G$ for all $g \in G$.*

Proof. Suppose that G is an ordered LA*-Γ-group. Furthermore, let $G\Gamma g = G$. As $g\Gamma G \subseteq G$, let $g \in G$. Then for all $s \in G$, we have $g = e\alpha g = (s^{-1}\beta s)\gamma g = s\delta(s^{-1}\theta g) \in s\Gamma G$ for $\alpha, \beta, \gamma, \delta, \theta \in \Gamma$. It follows that $G \subseteq s\Gamma G$. Hence $s\Gamma G = G$ for all $s \in G$. □

The following Theorem states no containment property of a proper relative ordered interior Γ-ideal in case an ordered LA-Γ-semigroup with left identity e is a left or a right relative ordered simple.

Theorem 3.2. *Suppose that an ordered LA-Γ-semigroup $(G, \Gamma, \leq)$ with left identity e is a left or right relative ordered simple. Then no proper relative ordered interior Γ-ideal is contained in G.*

Proof. Suppose that G is a right relative ordered simple and $S \subseteq G$. It follows that $(s\Gamma S]_S = S$ for every $s \in S$. Suppose that R is a relative ordered right Γ-ideal of G.

Then $(r\Gamma S]_S = S$ for every $r \in R \subseteq S$. As $s \in S$ and $r \in R$. We have the following:

$$\begin{aligned} s \in (r\Gamma S]_S = (r\alpha(r\Gamma S]_S]_S &\subseteq (r\beta(r\Gamma S)_S]_S = (r\gamma((e\delta r)\Gamma S)]_S \\ &= (r\theta((S\Gamma r)\gamma_1 e)]_S \\ &= ((S\Gamma r)\Gamma(r\gamma_2 e)]_S \\ &\subseteq ((S\Gamma R)\Gamma S]_S \\ &\subseteq (R]_S = R, \end{aligned}$$

for $\alpha, \beta, \gamma, \delta, \theta, \gamma_1, \gamma_2 \in \Gamma$, since every relative ordered right Γ-ideal of G is a relative ordered interior Γ-ideal of G, and also that every relative ordered right Γ-ideal of G is a relative ordered Γ-ideal of G by Lemma 2.1. Therefore $S \subseteq R$. Hence $R = S$. Similarly, it is for relative ordered left simple. □

Theorem 3.3. *Suppose that an ordered LA-Γ-semigroup G is both a left and a right relative ordered simple. Then no proper relative ordered bi-Γ-ideal is contained in G.*

Proof. Suppose that G is a left as well as a right relative ordered simple, and $S \subseteq G$. Then $(S\Gamma s]_S = S$ and $(s\Gamma S]_S = S$ for every $s \in S$. Suppose that I is a relative ordered Γ-ideal of G. It follows that I is a left and a right relative ordered Γ-ideal of G. So, $(S\Gamma i]_S = S$ and $(i\Gamma S]_S = S$ for every $i \in I \subseteq S$. Since $s \in S$ and $i \in I$, we have the following:

$$s \in (S\Gamma i]_S = ((i\Gamma S]_S\Gamma i]_S \subseteq ((i\Gamma S)\Gamma i] \subseteq ((I\Gamma S)\Gamma I]_S \subseteq (I]_S = I,$$

for every relative ordered Γ-ideal of G is a relative ordered bi-Γ-ideal of G by Lemma 2.4. Therefore $S \subseteq I$. Hence $I = S$. □

Theorem 3.4. *An ordered LA-Γ-semigroup G with left identity e is a relative ordered left and a right simple if and only if it contains no proper relative ordered quasi-Γ-ideal.*

Proof. $\Rightarrow$ Suppose that G is a relative ordered left and a right simple, and $S \subseteq G$. Then we have $(S\Gamma g]_S = G$ and $(g\Gamma S]_S = G$ for every $g \in S$. Further suppose that I is a relative ordered Γ-ideal of G. It then follows that I is a left and a relative ordered right Γ-ideal of G. Therefore $(S\Gamma i]_S = S$ and $(i\Gamma S]_S = S$ for every $i \in I \subseteq S$. Since $s \in S$ and $i \in I$, we have the following:

$$s \in (i\Gamma S]_S \cap (S\Gamma i]_S \subseteq (I\Gamma S]_S \cap (S\Gamma I]_S \subseteq I,$$

since I is relative ordered quasi-Γ-ideal of G, and also for every relative ordered Γ-ideal of G is a relative ordered quasi-Γ-ideal of G by Lemma 2.5. Hence $S = I$.

$\Leftarrow$ Suppose that G contains no proper relative ordered quasi-Γ-ideal. Therefore G contains neither any proper relative ordered left Γ-ideal nor any proper relative ordered right Γ-ideal for, every left and every right relative ordered Γ-ideal of G is a relative ordered quasi Γ-ideal of G by Lemma 2.5. Suppose that $s \in S$. We need to prove that

$(S\Gamma s]_S$ is a relative ordered left Γ-ideal of G, and also that $(s\Gamma S]_S$ is a relative ordered right Γ-ideal of G. We have the following:

$$\begin{aligned} S\Gamma(S\Gamma s]_S &\subseteq (S]_S\Gamma(S\Gamma s]_S \\ &\subseteq (S\Gamma(S\Gamma s)]_S = ((S\Gamma e)\Gamma(S\Gamma s)]_S \\ &= ((S\Gamma S)\Gamma(e\alpha s)]_S \\ &\subseteq (S\Gamma s]_S, \end{aligned}$$

and $((S\Gamma s]_S]_S \subseteq (S\Gamma s]_S$ for $\alpha \in \Gamma$. Furthermore, we have the following:

$$\begin{aligned} (s\Gamma S]_S\Gamma S \subseteq (s\Gamma S]_S\Gamma(S]_S &\subseteq ((s\Gamma S)\Gamma S]_S \\ &= ((s\Gamma S)\Gamma(e\Gamma S)]_S \\ &= ((s\alpha e)\Gamma(S\Gamma S)]_S \\ &\subseteq ((s\beta e)\Gamma S]_S \\ &= (e\gamma(s\Gamma S)]_S = (s\Gamma S]_S, \end{aligned}$$

in an ordered LA-Γ-semigroup for $\alpha, \beta, \gamma \in \Gamma$, and also that, we have $((s\Gamma S]_S]_S \subseteq (s\Gamma S]_S$. Therefore $(S\Gamma s]_S$ is a relative ordered left Γ-ideal of G and $(s\Gamma S]_S$ is a right relative ordered Γ-ideal of G.

Clearly, we have $(S\Gamma s]_S \subseteq S$ and $(s\Gamma S]_S \subseteq S$. Therefore, we have the following: $s = e\alpha s \in S\Gamma s \subseteq (S\Gamma s]_S$, and

$$s = e\beta s = (e\gamma e)\delta s = e\gamma_1(s\gamma_2 e) = s\gamma_3 e \in s\Gamma S \subseteq (s\Gamma S]_S$$

in an ordered LA-Γ-semigroup for $\alpha, \beta, \gamma, \delta, \gamma_1, \gamma_2, \gamma_3 \in \Gamma$. It follows that $S \subseteq (S\Gamma s]_S$ and $S \subseteq (s\Gamma S]_S$. Therefore $(s\Gamma S]_S = (S\Gamma s]_S = S$ for all $s \in S$. Hence G is a relative ordered left simple as well as a relative ordered right simple. □

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