SHEAF REPRESENTATION OF ALMOST BOOLEAN ALGEBRAS

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Abstract

An Almost Boolean Algebra $(A, \wedge, \vee, 0)$ (abbreviated as ABA) is an Almost Distributive Lattice (ADL) with a maximal element in which for any $x \in A$, there exists $y \in A$ such that $x \wedge y = 0$ and $x \vee y$ is a maximal element in A. If (S, Π, X) is a sheaf of nontrivial discrete ADLs over a Boolean space such that for any global section f, support of f is open, then it is proved that the set $\Gamma(X, S)$ of all global sections is an ABA. Conversely, it is proved that every ABA is isomorphic to the ADL of global sections of a suitable sheaf of discrete ADL over a Boolean space.

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1 Introduction

The axiomatization of George Boole's two valued propositional calculus led to the concept of Boolean algebra (ring). M. H. Stone [3] established a strong duality between Boolean algebras (rings) and Compact Hausdorff topological spaces in which closed and open subsets form a base for the topology. After the notion of Boolean algebra (ring) came to light, several generalizations have come up in which the ring theoretic generalizations like regular rings, p-rings, biregular rings etc and the lattice theoretic generalizations like distributive lattices, Hayting algebras, post algebras, pseudo-complemented distributive lattices, Stone lattices etc. U. M. Swamy and G. C. Rao [6] have introduced a common abstraction of these two streams in the form of an Almost Distributive Lattice (ADL) as an algebra $(A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ which satisfies all the axioms of a distributive lattice with 0 except the commutativity of the binary operations and, in this case, the commutativity of either of the binary operation is equivalent to that of the other. An ADL A with a maximal element m is said to be an ABA if for each $x \in A$, there exists $y \in A$ such that $x \wedge y = 0$ and $x \vee y$ is maximal (equivalently, for each $x \in A$, the interval $[0, x]$ is a Boolean algebra under the induced operations $\wedge$ and $\vee$).

In this paper, we have considered the prime spectrum of a nontrivial Almost Bololean Algebra which forms a compact, Hausdorff and totally disconnected topological space and it is well known that such spaces are called Boolean space. With this as the base space, we have constructed a sheaf of ADLs whose ADL of continuous sections is isomorphic to the given Almost Boolean Algebra.

2 Preliminaries

In this section, we collect certain definitions and properties of ADLs from [6, 7, 8] that are required in the main text of this paper.

Definition 2.1. An algebra $A = (A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL) if it satisfies the following identities.

(1) $0 \wedge a = 0$

(2) $a \vee 0 = a$

(3) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(4) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(5) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(6) $(a \vee b) \wedge b = b$.

Example 2.2. Every non-empty set A can be regarded as an ADL as follows. Let $a_0 \in X$. Define the binary operations $\vee, \wedge$ on X by

$$a \vee b = \begin{cases} a & \text{if } a \neq a_0 \\ b & \text{if } a = a_0 \end{cases} \qquad a \wedge b = \begin{cases} b & \text{if } a \neq a_0 \\ a_0 & \text{if } a = a_0. \end{cases}$$

Then $(A, \vee, \wedge, a_0)$ is an ADL (where a_0 is the zero element).

Definition 2.3. Let $(A, \wedge, \vee, 0)$ be an ADL. For any a and $b \in A$, define

$$a \leq b \; if \; a = a \wedge b \text{ (equivalently } a \vee b = b).$$

Then $\leq$ is a partial order on A.

Theorem 2.4. *If $(A, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in A$, we have the following:*

(1) $a \vee b = a \Leftrightarrow a \wedge b = b$

(2) $a \vee b = b \Leftrightarrow a \wedge b = a$

(3) $\wedge$ *is associative in* A

(4) $a \wedge b \wedge c = b \wedge a \wedge c$

(5) $(a \vee b) \wedge c = (b \vee a) \wedge c$

(6) $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$

(7) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(8) $a \wedge (a \vee b) = a, \quad (a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$

(9) $a \leq a \vee b$ *and* $a \wedge b \leq b$

(10) $a \wedge a = a$ *and* $a \vee a = a$

(11) $0 \vee a = a$ *and* $a \wedge 0 = 0$

(12) $If \; a \leq c, \quad b \leq c$ *then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$

(13) $a \vee b = (a \vee b) \vee a$.

Definition 2.5. A homomorphism between ADLs $(A, \vee, \wedge, 0)$ into an ADL L', we mean, a mapping $f : A \to A'$ satisfying the following:

(1) $f(a \vee b) = f(a) \vee f(b)$

(2) $f(a \wedge b) = f(a) \wedge f(b)$

(3) $f(0) = 0$.

A nonempty subset I of an ADL A is called an ideal of A if $x \vee y \in I$ and $x \wedge a \in I$ whenever $x, y \in I$ and $a \in L$. For any $X \subseteq A$, the ideal generated by X is $(X] = \big\{\big(\bigvee_{i=1}^{n} a_i\big) \wedge x : a_i \in X, x \in A, n \in \mathbb{Z}^+\big\}$. If $X = \{x\}$, then we write $(x]$ for $(X]$ and this is called a principal ideal generated by x. The set of all principal ideals of A is a distributive lattice and it is denoted by $P\mathfrak{I}(A)$. A proper ideal P of A is called prime if for any $x, y \in A, x \wedge y \in P$ then $x \in P$ or $y \in P$. For any $x, y \in A$ with $x \leq y, [x, y] = \{t \in A : x \leq t \leq y\}$ is a bounded distributive lattice with respect to the operations induced from those on A. An element m is maximal in $(A, \leq)$ if and only if $m \wedge x = x$ for all $x \in X$. An ADL A is said to be discrete if every nonero element is maximal. The ADL given in the example 2.2 is a discrete ADL. For any $X \subseteq A,\ X^* = \big\{a \in A :\ x \wedge a = 0\ \forall\ x \in X\big\}$ is an ideal of A and X^* is called the annihilator of X. We don't know, so far, whether $\vee$ is associative in an ADL or not. In this paper A denotes an ADL in which $\vee$ is associative.

Lemma 2.6. *Let A be an ADL and I is an ideal of A. Then, for any $a, b \in A$, we have the following:*

(1) $(a] = \big\{a \wedge x : x \in A\big\}$

(2) $a \in (b] \Leftrightarrow b \wedge a = a$

(3) $a \wedge b \in I \Leftrightarrow b \wedge a \in I$

(4) $(a] \cap (b] = (a \wedge b] = (b \wedge a]$

(5) $(a] \vee (b] = (a \vee b] = (b \vee a]$

(6) $(a] = A \Longleftrightarrow a$ *is maximal.*

Lemma 2.7. *Let A be an ADL and $x, y \in A$. Then the following statements hold:*

(1) $\{x \vee y\}^* = \{x\}^* \cap \{y\}^*$

(2) $\{x \wedge y\}^* = \{y \wedge x\}^*$

(3) $\{x\}^{***} = \{x\}^*$

(4) $x \leq y \Rightarrow \{y\}^* \subseteq \{x\}^*$

(5) $\{x \wedge y\}^{**} = \{x\}^{**} \cap \{y\}^{**}$

Definition 2.8. An ADL $(A, \wedge, \vee, 0)$ is said to be relatively complemented if every interval in A is a Boolean algebra.

Theorem 2.9. *Let A be an ADL. Then the following are equivalent to each other.*

(1) For any $a, b \in A$ there exists $x \in A$ such that $a \wedge x = 0$ and $a \vee x = a \vee b$

(2) For any $a \leq b$ in A, $[a, b]$ is a complemented lattice.

(3) For any $a \in A$, $[0, a]$ is complemented lattice

Definition 2.10. A nontrivial ADL A is called an Almost Boolean algebra (ABA) if it has a maximal element and satisfies one, and hence all of the equivalent conditions given in the above theorem.

Theorem 2.11. *Let A be an ADL with a maximal element. Then the following are equivalent to each other.*

(1) A is an Almost Boolean algebra

(2) For any $a \in A$, there exists $b \in A$ such that $a \wedge b = 0$ and $a \vee b$ is maximal.

(3) $[0, m]$ is a Boolean algebra for all maximal elements m

(4) There esists a maximal element m such that $[0, m]$ is a Boolean algebra.

Theorem 2.12. *Let A be an ADL and m and n be maximal elements in A. Then the lattices $[0, m]$ and $[0, n]$ are isomorphic to each other. Moreover, the Boolean algebras $[0, m]$ and $[0, n]$ are isomorphic when A is Almost Boolean algebra.*

Theorem 2.13. *Let $(A, \wedge, \vee, 0)$ be an ABA. Then for any a and b in A there exists a unique $x \in A$ such that $a \wedge x = 0$ and $a \vee x = a \vee b$*

Definition 2.14. A nontrivial ADL A is called dense if $a \wedge b \neq 0$ for all $a \neq 0$ and $b \neq 0$ (equivalently, $\{a\}^* = \{0\}$, for any $0 \neq a \in A$).

3 Sheaf Representation of ABAs

Definition 3.1. A triple (S, Π, X) is called a sheaf over X if S and X are topological spaces and $\Pi : S \to X$ is a local homeomorphism of S onto X; that is Π is a surjective map and, for any $s \in S$, there are open sets G and U containing s and $\Pi(s)$ in S and X respectively such that the restriction of Π to G is a homeomorphism of G onto U.

Let (S, Π, X) be a sheaf over X. Then $\Pi : S \to X$ is continuous and open map. Here S is called the sheaf space, X the base space and Π the projection. For any $Y \subseteq X$, a continuous map $f : Y \to S$ such that $\Pi \circ f$ is identity on Y is called a section on Y. Sections on the whole space X are called global sections. For any $x \in X$, the set $S_x = \{s \in S : \Pi(s) = x\} = \Pi^{-1}(\{x\})$ is called the stalk at x. The set of all global sections will be denoted by $\Gamma(X, S)$.

Definition 3.2. A sheaf (S, Π, X) is called a sheaf of discrete ADLs if for each $x \in X$, the stalk S_x is an (discrete) ADL $(S_x, \wedge, \vee, 0_x)$ satisfies the following.

(1) the operations $\wedge$ and $\vee$ are continuous mappings of $S \otimes S$ into S, where $S \otimes S = \{(s, t) \in S \times S : \Pi(s) = \Pi(t)\}$

(2) the map $x \mapsto 0_x$ is a continuous map of X into S, where 0_x is the zero element in the stalk S_x

Theorem 3.3. *Let (S, Π, X) be a sheaf of ADLs. Then $\Gamma(X, S)$ is an ADL under the pointwise operations.*

Proof. Since $x \mapsto 0_x$ is continuous, it is a section on X and we denote it by $\overline{0}$. Therefore $\overline{0} \in \Gamma(X, S)$ and hence $\Gamma(X, S) \neq \emptyset$. For any $f, g \in \Gamma(X, S)$, we define $f \wedge g$ and $f \vee g$ by

$$(f \wedge g)(x) = f(x) \wedge g(x)$$
$$\text{and} \quad (f \vee g)(x) = f(x) \vee g(x) \text{ for all } x \in X.$$

The $\wedge$ and $\vee$ on the right-hand side are those in the stalk S_x which is an ADL. It can be easily verified that $\left(\Gamma(X, S), \wedge, \vee, \overline{0}\right)$ is an ADL. □

Let us recall that a topological space X is called a Boolean space if X is Compact, Hausdorff and totally disconnected. Now, we first find an ABA from a given sheaf over a Boolean space.

Theorem 3.4. *Let (S, Π, X) be a sheaf of nontrivial discrete ADLs over a Boolean space X such that, for any global section f, support $|f| = \{x \in X : f(x) \neq 0_x\}$ is open in X. Then $\Gamma(X, S)$ is an ABA.*

Proof. By the above theorem $\Gamma(X, S)$ is an ADL under the pointwise operations. It is easily verified that, for any global sections f and g, the set

$$\langle f, g\rangle = \{x \in X : f(x) = g(x)\}$$

is a closed set in X and by hypothesis $\langle f, \overline{0}\rangle(= |f|\ )$ is open also. Thus $|f|$ is a closed set in X for all global sections f. For each $x \in X$, we can choose $s(\neq 0_x) \in S_x$ and a global section f_x such that $f_x(x) = s \in S_x$ and hence $x \in |f_x|$, the support of f_x. Since to each $x \in X$, $|f_x|$ is clopen and X is compact, there exists clopen sets $U_1, U_2, \ldots U_n$ such that

$$U_i \cap U_j = \emptyset \text{ for } i \neq j$$

and $$U_1 \cup U_2 \cup \cdots \cup U_n = X$$

and there is global section f_i such that $f_i(y) \neq o_y$ for all $y \in U_i$.
Let $f = f_1 \vee f_2 \vee \cdots \vee f_n$. Then f is a global section and $|f| = X$. Let $g \in \Gamma(X, S)$. Then $(f \wedge g)(x) = f(x) \wedge g(x)$ for all $x \in X$ (since each S_x is discrete and hence every nonzero element in S_x is maximal) and hence $f \wedge g = g$ for all $g \in \Gamma(X, S)$. Therefore $\Gamma(X, S)$ is an ADL with a maximal element. Further, let $g \in \Gamma(X, S)$ and f be any maximal element in $\Gamma(X, S)$. Then $f(x) \neq 0_x$ for all $x \in X$. Define $h : S \to X$ by

$$h(x) = \begin{cases} 0_x & \text{if } \ x \in |g| \\ f(x) & \text{if } \ x \notin |g|. \end{cases}$$

Then h is continuous (since $x \mapsto 0_x$ and f are continuous and $|g|$ is clopen). Therefore $h \in \Gamma(X, S)$. Also $(g \wedge h)(x) = g(x) \wedge h(x) = 0_x$ for all $x \in X$ and hence $g \wedge h = \overline{0}$. Further $(g \vee h)(x) \neq 0_x$ for all $x \in X$ and hence $g \vee h$ is maximal in $\Gamma(X, S)$. Thus $\Gamma(X, S)$ is an ABA. □

In order to prove a converse of above theorem, we first obtain a sheaf over a Boolean space from a given ABA. First we recall the following from [6].

Definition 3.5. For any element a in an ADL A,
define

$$\theta_a = \big\{(x, y) \in A \times A : a \wedge x = a \wedge y\big\}.$$

Then θ_a is a congruence relation on A.

Lemma 3.6. *Let A be an ABA and P be a prime ideal of A. Let*

$$\theta_P = \big\{(x, y) \in A \times A : a \wedge x = a \wedge y \ forsome\ a \in A - P\big\}.$$

Then θ_P is a congruence relation on A and the Quotient A/θ_P is a nontrivial discrete ABA.

Proof. Clearly $\theta_P = \bigcup_{a\in A-P} \theta_a$ and it is easy to verified that θ_P is a congruence relation on A. Also, for any $x \in A$,

$$x \in P \Longleftrightarrow (x, 0) \in \theta_P.$$

Since P is a proper ideal of A, there exists $x \in A-P$ and hence $(x, 0) \notin \theta_P$ so that $x/\theta_P \neq 0/\theta_P$. Therefore A/θ_P is a nontrivial ADL. Now, if $x/\theta_P \neq 0/\theta_P$, then $x \notin P$ and $(x \wedge a, a) \in \theta_P$ so that $x/\theta_P \wedge a/\theta_P = (x \wedge a)/\theta_P = a/\theta_P$ for all $a \in A$ and hence x/θ_P is maximal in A/θ_P. Therefore, A/θ_P is discrete ABA. □

Let us recall from [6] that, for any ABA $A = (A, \wedge, \vee, 0, m)$, $Spec(A)$ denotes the space of all prime ideals of A together with the hull-kernal topology for which $\{X_a : a \in A\}$ is a base, where $X_a = \{P \in Spec(A) : a \notin P\}$ and that $Spec(A)$ is a Boolean space (a Compact Hausdorff and totally disconnected topological space).

Lemma 3.7. *Let A be an ABA and $a \in A$. Then*
$$\theta_a = \bigcap\big\{\theta_P : P \in Spec(A) \text{ and } a \notin P\big\} = \bigcap\big\{\theta_P : P \in X_a\big\}.$$

Proof. First, it can be easily verified $x, y \in A$, the set $\langle x, y\rangle = \{b \in A : b \wedge x = b \wedge y\}$ is an ideal of A. Let $a \in A$. we have that $\theta_P = \bigcup_{b\in A-P} \theta_b$. Therefore

$$P \in X_a \Rightarrow a \notin P \Rightarrow \theta_a \subseteq \theta_P.$$

Hence $\theta_a \subseteq \bigcap_{P\in X_a} \theta_P$. On the otherhand, suppose that $(x, y) \notin \theta_a$. Then $a \wedge x \neq a \wedge y$ and hence $a \notin \langle x, y$. Since $\langle x, y\rangle$ is an ideal, there exists a prime ideal P of A such that $\langle x, y\rangle \nsubseteq P$ and $a \notin P$. Now $P \in X_a$ and $(x, y) \notin \theta_P$. Therefore $\bigcap_{P\in X_a} \theta_P \subseteq \theta_a$. Thus $\bigcap_{P\in X_a} \theta_P = \theta_a$. □

Corollary 3.8. *If A is an ABA, then $\bigcap\big\{\theta_P : P \in Spec(A)\big\} = \Delta_A$, the diagonal relation on A.*

Proof. It follows by the fact that, for any maximal element m in A, $X_m = X$ and, $\theta_m = \Delta_A$. □

Theorem 3.9. *Any ABA is isomorphic to $\Gamma(X, S)$ for a suitable sheaf (S, Π, X) of discrete ADLs over a Boolean space X.*

Proof. Let A ba an ABA and $X = Spec(A)$. Then X is a Boolean space. For any $P \in X$, let $S_P = A/\theta_P$ and S be the disjoint union of $A/{\theta_P}'^s$, $P \in X$. For any $x \in A$, define $\hat{x} : X \to S$ by $\hat{x}(P) = x/\theta_P$. Define $\Pi : S \to X$ by

$\Pi(s) = P$ if $s \in S_P$. Let S be equipped with the largest topology with respect to which each $\hat{x}$, $x \in A$, is continuous. Now we shall prove that (S, Π, X) is a sheaf of discrete ADLs over the Boolean space X and finally we proved that $A \cong \Gamma(X, S)$. First, we prove that the class $\{\hat{x}(X_a) : x, a \in A\}$ forms a base for the topology on S. Clearly $\bigcup_{x,a\in A} \hat{x}(X_a) = S$. For any $x, y \in A$ and $a, b \in A$, we have

$$\begin{aligned} s \in \hat{x}(X_a) \cap \hat{y}(X_b) &\Rightarrow s = x/\theta_P = y/\theta_Q \text{ for some } P \in X_a, Q \in X_b \\ &\Rightarrow P = \Pi(s) = Q \\ &\Rightarrow P = Q \in X_a \cap X_b \text{ and } c \wedge x = c \wedge y \text{ for some } c \in A - P \\ &\Rightarrow P \in X_{a\wedge b\wedge c} \text{ and } a \wedge b \wedge c \wedge x = a \wedge b \wedge c \wedge y \\ &\Rightarrow P \in X_{a\wedge b\wedge c} \text{ and } s = \hat{x}(P) = \hat{y}(Q) \\ &\Rightarrow s \in \hat{x}(X_{a\wedge b\wedge c}) \subseteq \hat{x}(X_a) \cap \hat{y}(X_b). \end{aligned}$$

Therefore the class mentioned is a base for a topology, say τ on S.
Also, for any x, y and $a \in A$,

$$\begin{aligned} P \in \hat{x}^{-1}(\hat{y}(X_a)) &\Rightarrow x/\theta_P = y/\theta_P \text{ and } P \in X_a \\ &\Rightarrow b \wedge x = b \wedge y \text{ for some } b \in A - P \\ &\Rightarrow P \in X_{a\wedge b} \subseteq \hat{x}^{-1}(\hat{y}(X_a)). \end{aligned}$$

Therefore each $\hat{x}$, $x \in A$ is continuous with respect to the topology τ and it can be easily proved that τ is the largest topology on S with respect to which each $\hat{x}$ is continuous. Since $\Pi(s) = P$ for all $s \in S_P = A/\theta_P, P \in X$, $\Pi : S \to X$ is a bijection mapping. Let $s \in S$. Then $s = x/\theta_P = \hat{x}(P)$ for some $P \in X$ and $x \in A$. If $\Pi(s) = P$, choose $a \in A - P$, then $P \in X_a$ and $s \in \hat{x}(X_a)$. Since $\hat{x}(X_a)$ is open in S and $\hat{x}(X_a)$ is a neighbourhood of s in S and X_a is open neighbourhood of $\Pi(s)$ in X and hence it can ba easily verified that any restriction of Π to $\hat{x}(X_a)$is a homeomorphism of $\hat{x}(X_a)$ onto X_a. Therefore Π is a local homeomorphism of S onto X and hence (S, Π, X) is a sheaf over Boolean space X. Since θ_P is a congruence on A, it follows that the continuity of ADL operations $\wedge$, $\vee$ and 0. And, for any $x, y \in A$, $\{P \in X : (x, y) \in \theta_P\}$ is an open subset of X. Also, for each $P \in X, A/\theta_P$ is a nontrivial discrete ADL (by lemma 3.6). Thus (S, Π, X) is a sheaf of nontrivial discrete ADLs over the Boolean space X. Since the operations are point-wise and each stalk A/θ_P is an ADL, it follows that $\Gamma(X, S)$ is also an ADL. Finally define $\alpha : A \to \Gamma(X, S)$ by $\alpha(x) = \hat{x}$ for all $x \in A$. Since $\widehat{x \wedge y} = \hat{x} \wedge \hat{y}$ and $\widehat{x \vee y} = \hat{x} \vee \hat{y}$, it follows that α is a homomorphism of ADLs. Also, for any $x, y \in A$,

$$\begin{aligned} \alpha(x) = \alpha(y) &\Rightarrow \hat{x} = \hat{y} \\ &\Rightarrow \hat{x}(P) = \hat{y}(P) \text{ for all } P \in X \\ &\Rightarrow x/\theta_P = y/\theta_P \text{ for all } P \in X \\ &\Rightarrow (x, y) \in \bigcap_{P\in X} \theta_P = \Delta_A \\ &\Rightarrow x = y. \end{aligned}$$

Therefore α is an injection. Lastly, let $f \in \Gamma(X, S)$. For each $P \in X$, there exists $x_P \in A$ such that $f(P) = x_P/\theta_P = \hat{x_P}(P)$. Since f and $\hat{x_P}$ are sections on X, we can choose open sets G and W in S and X respectively such that $f(P)(= \hat{x_P}(P)) \in G$, $P(= \Pi(f(P))) \in W$ and $\Pi/G : G \to W$ is a homeomorphism, $f^{-1}(G) \cap \hat{x_P}^{-1}(G)$ is open in X and hence $f^{-1}(G) \cap \hat{x_P}^{-1}(G) = X_{a_P}$ for some $a_P \in A$ (since X_a's, $a \in A$ form a base for open sets) and $P \in X_{a_P}$. Also, for any $Q \in X_{a_P}, f(Q)$ and $\hat{x}_P(Q) \in G$ and $\Pi(f(Q)) = \Pi(\hat{x_P}(Q))$ and implies that $f(Q) = \hat{x_P}(Q)$ (since Π is an injection on G). By using the compactness of X and the fact that X in a Boolean space, there exists $a_1, a_2, \cdots a_n \in A$ and $x_1, x_2, \cdots x_n \in A$ such that $\bigcup_{i=1}^{n} x_i = X$ and $f(Q) = \hat{x}_i(Q)$ for all $Q \in X_{a_i}$. Since each X_{a_i} is clopen, we choose $b_1, b_2, \cdots b_n \in A$ such that

$$X_{b_1} = X_{a_1}$$
$$X_{b_2} = x_{a_2} \cap (X - X_{a_1})$$
$$X_{b_3} = X_{a_3} \cap (X - X_{a_1}) \cap (X - X_{a_2})$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$X_{b_n} = X_{a_n} \cap \bigcap_{i=1}^{n-1} (X - X_{a_i}).$$

Then $X_{b_i \wedge b_j} = X_{b_i} \cap X_{b_j} = \emptyset$ for all $i \neq j$, $\bigcup_{i=1}^{n} X_{b_i} = X$ and $f(Q) = \hat{x}_i(Q)$ for all $Q \in X_{b_i}$. Put $x = \bigvee_{i=1}^{n} (b_i \wedge x_i)$. Then, for any $1 \leq i \leq n$, $(b_i \wedge x_i, x_i) \in \theta_{b_i}$ and hence $(b_i \wedge x_i) \in \theta_Q$ for all $Q \in X_{b_i}$ and it implies that $(b_i \wedge x_i)/\theta_Q = x_i/\theta_Q$ so that $\widehat{b_i \wedge x_i}(Q) = \hat{x}_i(Q) = f(Q)$ for all $Q \in X_{b_i}$ and for $j \neq i$, $\widehat{b_j \wedge x_j}(Q) = \hat{b_j}(Q) \wedge \hat{x_j}(Q) = \hat{0}(Q)$ for all $Q \in X_{b_j}$ (since $b_j \in Q \Rightarrow \hat{b_j}(Q) = \hat{0}(Q)$). Therefore, for any $Q \in X$, there exists unique i such that $Q \in X_{b_i}$ and $Q \notin X_{b_j}$ for all $j \neq i$ and hence $\hat{x}(Q) = \bigvee_{i=1}^{n} \widehat{(b_i \wedge x_i)}(Q) = \hat{x}_i(Q) = f(Q)$. Thus $f = \hat{x}$. This implies that $\alpha : A \to \Gamma(X, S)$ is an isomorphism of A onto $\Gamma(X, S)$. □

Note that in the above sheaf (S, Π, X) the support of any global section is a closed set, since any global section f is of the form $\hat{x}$ for same $x \in A$ and $|f| = |\hat{x}| = X_x$ which is clopen in X.

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