A Brief Review About Fractional Calculus

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Abstract

In this paper, we recall some basic facts and preliminaries related to fractional calculus by making a literature review for its definitions and properties. This would provide sufficient knowledge about this important topic. Several lemmas and theorems are shown in detail for completeness.

Key-words: Fractional Calculus, Fractional Derivative, Gamma function, Beta function, Binomial formula, Taylor's expansion.

1. Introduction

Fractional calculus indicates the integration or the differentiation of non-integer order. Interestingly enough, this topic has a long history in calculus. The first discussion of fractional calculus was between Leibniz and L'Hopital. He actually asked the latter about the differentiation of order half of certain functions. However, there are some mathematicians, like Riemann, Abel, Liouville, and Lacroix, who laid the foundations for fractional calculus and dominated the field.

Fractional Calculus is the branch of calculus that generalizes the derivative of a function to non-integer order, allowing calculations such as deriving a function to $1/_2$ order. Despite "generalized" would be a better option, the name "fractional" is used for denoting this kind of derivative.

Note : These notes are not a summary of standard method or an academic text, so some conventional developments are omitted while some other are introduced . The same happens with notation.

The derivative of a function f is defined as

$$D^1 f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad 1.1$$

Iterating this operation yields an expression for the nth derivative of a function. As can be easily seen and proved by induction for any natural number n,

1.2

$$D^n f(x) = \lim_{h \to 0} h^{-n} \sum_{m=0}^{n} (-1)^m_x \binom{n}{m} f(x + (n-m)h)$$

Where

1.3

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$$

Or equivalently,

1.4

$$D^n f(x) = \begin{matrix} limh^{-n} \\ h \to 0 \end{matrix} \sum_{m=0}^{n} (-1)^m \binom{n}{m} f(x - mh)$$

The case of $n = 0$ can be included as will.

Such an expression could be valuable for instance in a simple program for plotting the n-dt derivative of a function.

Viewing this expression one asks immediately if it can be generalized to any non-integer , real or complex number n. There are some reasons that can make us think so,

1. The fact that for any natural number n the calculation of the n-st derivative is given by an explicit formula (1.2) or (1.4).

2. That the generalization of the factorial by the gamma function allows

1.5

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!} = \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)}$$

Which also is valid for non-integer values.

3. The likeness of (1.2) to the binomial formula

1.6

$$(a+b)^n = \sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m$$

Which can be generalized to any complex number α by

1.7

$$(a+b)^\alpha = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{n!\,(\alpha-n+1)} a^{\alpha-n} b^n$$

Which is convergent if

1.8

$$|b| < a$$

There are some desirable properties that could be required to the fractional derivative,

Existence and continuity form times derivable functions, for any n which modulus is equal or than m.

For n = 0 the result should be the function itself; for n > 0 integer values it should be equal to the ordinary derivative and for n < 0 integer values it should be equal to the ordinary derivative and for n < 0 integer values it should be equal to ordinary integration – regardless the integrations constant.

Iterating should not give problems,

1.9

$$D^{\alpha+\beta}f(x) = D^{\alpha}D^{\beta}f(x)$$

Linearity,

1.10

$$D^{\alpha}\big(af(x)+bg(x)\big) = aD^{\alpha}f(x)+bD^{\alpha}g(x)$$

Allowing Taylor's expansion in some other way.

Its characteristic property should be preserved for the exponential function,

1.11

$$D^{\alpha}e^{x} = e^{x}$$

2. Exponentials

The case of the exponential function is specially simple and gives some clues about the generalization of the derivatives. Following (1.2),

$$D^{\alpha}e^{ax} = \begin{matrix} lim h^{-\alpha} \\ h\to 0\end{matrix}\sum_{n=0}^{\alpha}(-1)^n\binom{\alpha}{n}e^{\alpha(x+(\alpha-n)h)}$$

$$= e^{ax}\begin{matrix} lim h^{-\alpha} \\ h\to 0\end{matrix}\sum_{n=0}^{\alpha}(-1)^n\binom{\alpha}{n}(e^{ah})^{\alpha-n}$$

$$e^{ax}\begin{matrix} lim h^{-\alpha} \\ h\to 0\end{matrix}(e^{ah}-1)\alpha$$

$$= a^{\alpha}e^{ax}$$

2.1

The above limit exists for any complex number α. However, it should be noted that in the substitution of the binomial formula a nature number has been considered. We shall deal with this problem later to get our first generalization of the derivative. Applying this to the imaginary unit, 2.2

$$D^{\alpha}\cos(x) + i\,D^{\alpha}\sin(x) = D^{\alpha}e^{ix} = i^{\alpha}e^{ix}$$

$$= e^{\frac{\alpha\pi i}{2}}e^{ix} = e^{i\left(x+\frac{\alpha\pi}{2}\right)}$$

$$= cos\left(x+\frac{\alpha\pi}{2}\right) + isin\left(x+\frac{\alpha\pi}{2}\right)$$

And 2.3

$$D^{\alpha}\cos(x) - iD^{\alpha}\sin(x) = D^{\alpha}e^{-ix} = (-i)^{\alpha}e^{-ix}$$

$$= e^{\frac{-\alpha\pi i}{2}}e^{-ix} = e^{-i\left(x+\frac{\alpha\pi}{2}\right)}$$

$$= cos\left(x+\frac{\alpha\pi}{2}\right) - isin\left(x+\frac{\alpha\pi}{2}\right)$$

Solving this system we have the next definition for the sine and cosine derivatives.

2.4

$$D^{\alpha}\sin(x) = sin\left(x+\frac{\alpha\pi}{2}\right)$$

And

2.5

$$D^{\alpha}\sin(x) = cos\left(x + \frac{\alpha\pi}{2}\right)$$

We could expect expect these relations for the sine and cosine derivatives to be maintained in the generalization of the derivative. Applying the above method we also can calculate the following,

2.6

$$D^{\alpha}\cos(ax) + iD^{\alpha}\sin(ax) = D^{\alpha}e^{i\alpha x} = (ai)^{\alpha x}e^{i\alpha x}$$
$$= a^{\alpha}e^{\frac{\alpha\pi i}{2}}e^{i\alpha x} = a^{\alpha}e^{i\left(ax+\frac{\alpha\pi}{2}\right)}$$
$$= a^{\alpha}cos\left(ax + \frac{\alpha\pi}{2}\right) + ia^{\alpha}sin\left(ax + \frac{\alpha\pi}{2}\right)$$

And

2.7

$$D^{\alpha}\cos(ax) - iD^{\alpha}\sin(ax) = D^{\alpha}e^{i\alpha x} = (-ai)^{\alpha x}e^{-i\alpha x}$$
$$= a^{\alpha}e^{\frac{\alpha\pi i}{2}}e^{-i\alpha x} = a^{\alpha}e^{-i\left(ax+\frac{\alpha\pi}{2}\right)}$$
$$= a^{\alpha}cos\left(ax + \frac{\alpha\pi}{2}\right) - ia^{\alpha}sin\left(ax + \frac{\alpha\pi}{2}\right)$$

Thus 2.8

$$D^{\alpha}\sin(ax) = a^{\alpha}sin\left(ax + \frac{\alpha\pi}{2}\right)$$

And 2.9

$$D^{\alpha}\cos(ax) = a^{\alpha}cos\left(ax + \frac{\alpha\pi}{2}\right)$$

Indeed, the above result of the exponential can be applied to any function that can be expanded in exponentials 2.10

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$$
$$\Rightarrow D^{\alpha}f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}D^{\alpha}e^{inx} = \sum_{n=-\infty}^{\infty} a_n (ni)^{\alpha}e^{inx}$$
$$= \sum_{n=-\infty}^{\infty} a_n n^{\alpha}e^{\frac{\alpha\pi i}{2}}e^{inx} = \sum_{n=-\infty}^{\infty} a_n n^{\alpha}e^{i\left(nx \;\; \frac{\alpha\pi}{2}\right)}$$

Expanding the function in Fourier series, 2.11

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{e^{inx}}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int}dt$$

$$\Rightarrow D^{\alpha} f(x) = \sum_{n=-\infty}^{\infty} \frac{n^{\alpha} e^{i\left(nx+\frac{\alpha\pi}{2}\right)}}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int}dt$$

This method can be useful for calculating fractional derivatives of trigonometric functions.

3. Beta function

Definition :- The Beta function can be defined as follows:

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\, dx,$$

$$where\ p,q > 0$$

Theorem :- For $p, q \in \mathbb{R}^+$,we have: $\beta(p,q) = \beta(q,p)$.

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \frac{\Gamma(q)\Gamma(p)}{\Gamma(q+p)} = \beta(q,p)$$

Also For $p, q \in \mathbb{R}^+$, Show that:

$$\beta(p,q) = \int_0^{\infty} \frac{x^{p-1}}{(1+x)^{p+q}} dx$$

Proof. :- In order to prove this result, one can use first property, i.e.,

$$\beta(p,q) = \int_0^1 x^{q-1}\,(1-x)^{p-1}dx$$

By using the substitution $= \frac{1}{1+y}$, we can obtain:

$$\beta(p,q) = \int_{\infty}^{0} \frac{1}{(1+y)^{q-1}}\left(1-\frac{1}{1+y}\right)^{p-1}\left(\frac{-1}{(1+y)^2}\right)dy$$

This consequently implies:

$$\beta(p,q) = \int_0^{\infty} \frac{y^{p-1}}{(1+y)^{p+q}}\ dy.$$

Now, by replacing y instead of x, we obtain the wanted result.

4. Functions of the Derivative

It is worth to mention that being the expression of the $\alpha - st$ derivative of a function the derivative itself is 4.1

$$D = \lim_{h \to 0} \frac{d_h - 1}{h}$$

And being the $\alpha - st$ derivative or iterating the differentiation α times powering it to, applying other function to the derivative could be also considered. If the function applied to the derivative can be expanded in power of x, 4.2

$$g(x) = \sum_{n=-\infty}^{\infty} a_n x^n \Rightarrow g(D)f(x) = \sum_{n=-\infty}^{\infty} a_n D^n f(x)$$

The result is a weighted sum of different order derivatives. These functions of the derivative are usually known as "formal differential operators". As an example, the exponential of the derivative applied to the exponential would give the following result that could be valuable for calculating functions of the derivative when both f and g can be expanded in exponentials 4.3

$$e^{ax} \sum_{n=0}^{\infty} \frac{a^n x^n}{n!} \Rightarrow$$

$$\Rightarrow e^{aD} e^{bx} = \sum_{n=0}^{\infty} \frac{a^n D^n e^{bx}}{n!} = \sum_{n=0}^{\infty} \frac{a^n b^n e^{bx}}{n!} = e^{ab} e^{bx} = e^{b(x+a)}$$

If both function f and g can be expanded in positive powers of x.

$$g(x) \sum_{n=0}^{\infty} a_n x^n, f(x) \sum_{n=0}^{\infty} b_n x^n \Rightarrow$$

$$\Rightarrow e^{aD} e^{bx} = \sum_{n=0}^{\infty} \frac{a^n D^n e^{bx}}{n!} = \sum_{n=0}^{\infty} \frac{a^n b^n e^{bx}}{n!} = e^{ab} e^{bx} = e^{b(x+a)}$$

If both function f and g can be expanded in positive powers of x. 4.4

$$g(x) \sum_{n=0}^{\infty} a_n x^n, f(x) \sum_{n=0}^{\infty} b_n x^n \Rightarrow$$

$$\Rightarrow g(D)f(x) = \sum_{n=0}^{\infty} a_n D^n \sum_{m=0}^{\infty} b_m x^m = \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} b_m D^n x^m$$

$$= \sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} b_m \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$$

$$= \sum_{n=0}^{\infty} a_n \sum_{l=-n}^{\infty} b_{l+n} \frac{\Gamma(l+n+1)}{\Gamma(l+1)} x^l$$

$$= \sum_{n=0}^{\infty} a_n \sum_{l=0}^{\infty} b_{l+n} \frac{(l+n)}{(l!)} x^l$$

$$= \sum_{l=0}^{\infty} \frac{x^l}{l!} \sum_{n=0}^{\infty} (l+n)!\, a_n b_{l+n}$$

Thus, 4.5

$$g(D)f(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$c_n = \frac{1}{n!} \sum_{m=0}^{\infty} (n - m)!\, a_m b_{n+m}$$

At a First glance this seems not very interesting, but interesting properties could be hidden under this apparent mess. We shall return later to this point with the help of other tools.

Conclusion

By doing a study of the literature on the definitions and properties of fractional calculus, we are able to recollect some fundamental details and prerequisites in this work. This would give you enough information on this crucial subject. For completeness, several lemmas and theorems are illustrated in detail.

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