ON WEAK FORMS OF β-OPEN AND β-CLOSED FUNCTIONS IN SMOOTH FUZZY TOPOLOGICAL SPACES

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Abstract.

In this paper, we introduce and study two new classes of fuzzy functions by using the notions of r-fuzzy β-open sets and r-fuzzy β-closure operator called weakly r-fuzzy β-open and weakly r-fuzzy β-closed functions. The connections between these r-fuzzy functions and other existing r-fuzzy topological functions are studied.

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1. Introduction and Preliminaries.

Sˆostak [18] introduced the fuzzy topology as an extension of Chang's fuzzy topology [4] and developed in many directions [7, 8, 17]. In fuzzy topological spaces, a weaker forms of fuzzy continuity by many authors [1, 2, 3, 6, 20]. Kim and Park [9] introduced r-δ-cluster points and δ-closure operators in Sˆostak fuzzy topological spaces. Park et al. [11] introduced the concept of fuzzy semi-preopen sets which is weaker than any of fuzzy semi-open or fuzzy preopen sets. In 1968, Velicko [19] studied some new types of open sets called θ-open sets and δ-open sets. In 1963, Levine initiated a new type of open set called semi-open set [10]. In 1993, Raychaudhuri and Mukherjee defined δ-preopen sets [16] and δ-semiopen sets by Park [14]. In 2008, Caldas [5] obtained θ-semi-open sets. In 2006, Shafei introduced fuzzy θ-closed [21] and fuzzy θ-open sets. Recently, M.Palanisamy [12] & [13] introduced r-fuzzy Z * -Open Sets and r-fuzzy Z* -Continuity and r-fuzzy Z * -Open Sets and r-fuzzy Z* -Continuity sets in fuzzy topological spaces in the sense of Sˆostak's. In this paper, we introduce the concept of r-fuzzy β-

open sets and r-fuzzy β-closure operator called weakly r-fuzzy β-open and weakly r-fuzzy β-closed functions. The connections between these r-fuzzy functions and other existing r-fuzzy topological functions are studied. Also, discuss about some characterizations and properties of these notions.

Throughout this article, we denote nonempty sets by X, Y etc., $I = [0,1]$ and $I_0 = (0,1]$.. For $\alpha \in 1$, $\overline{\alpha}(x) = \alpha$, $\forall x \in X$ A fuzzy point x_t for $t \in I_0$ is an element of I^X such that

$$x_t(y) = \begin{cases} t & if \quad y \ is\ equal\ to \ x \\ 0 & if \quad y\ is\ not\ equal\ to\ x. \end{cases}$$

Let $P_t(X)$ denote the set of all fuzzy points in X. A fuzzy point $x_t \in \mu$ iff $t < \mu(x)$. $\mu \in I^X$ is quasi-coincident with ν, denoted by $\mu q \upsilon$, if $\exists x \in X$ such that $\mu(x) + \upsilon(x) > 1$.

If μ is not quasi-coincident with υ, we denoted $\mu \bar{q} \upsilon$. If A is a subset of X, we define the characteristic function χ_A on X by

$$\chi_t(x) = \begin{cases} 1 & if \quad x \in A \\ 0 & if \quad if \quad x \notin A. \end{cases}$$

All notations and definitions will be standard in the fuzzy set theory.

Definition 2.2. [17]

A function $\tau : I^X \to I$ is called a fuzzy topology on X if it satisfies the following conditions:

(1) $\tau(\bar{0}) = \tau(\bar{1}) = 1,$

(2) $\tau(\vee_{i \in \Gamma} v_i) \geq \wedge_{i \in \Gamma} \tau(v_i),$ for any $\{v_i\}_{i \in \Gamma} \subset I^X$,

(3) $\tau(v_1 \wedge v_2) \geq \tau(v_1 \wedge v_2)$, for any $\upsilon_1, \upsilon_2 \in I^X$.

The pair (X,τ) is called a fuzzy topological space or Sˆostak's fuzzy topological space or smooth topological space (for short, fts, sfts, sts).

Remark 2.3. [17]

Let (X,τ) be a fts. Then, for every $r \in I_0$, $\tau = \{\upsilon \in I^X : \tau(\upsilon) \geq r\}$ is a Change's fuzzy topology on X.

Theorem 2.4. [17]

Let (X,τ) be a sfts. Then for each $\mu \in I^X$, $r \in I_0$, we define an operator $C_\tau : I^X \times I_0 \to I^X$ as follows:

$C_\tau(\mu, r) = \wedge\{\upsilon \in I^X : \mu \leq \upsilon, \tau(\underline{1} - \upsilon) \geq r\}..$

For $\mu, \upsilon \in I^X$ and $r, s \in I_0$, the operator C_τ satisfies the following conditions:

(1) $C_\tau(\underline{0}, r) = \underline{0},$

(2) $\lambda \leq C_\tau(\lambda, r),$

(3) $C_\tau(\lambda, r) \vee C_\tau(\mu, r) = C_\tau(\lambda \vee \mu, r),$

(4) $C_\tau(\lambda, r) \leq C_\tau(\mu, s)$ if $r \leq s,$

(5) $C_\tau(C_\tau(\lambda, r), r) = C_\tau(\mu, r).$

Theorem 2.5. [21]

Let (X,τ) be a sfts. Then for each $\mu \in I^X$, $r \in I_0$, we define an operator $I_\tau : I^X \times I_0 \to I^X$ as follows:

$I_\tau(\mu, r) = \vee\{\upsilon \in I^X : \mu \leq \upsilon, \tau(\upsilon) \geq r\}..$

For $\mu, \upsilon \in I^X$ and $r, s \in I_0$, the operator C_τ satisfies the following conditions:

(1) $I_\tau(\underline{1}, r) = \underline{1},$

(2) $\lambda \geq I_\tau(\lambda, r)$,

(3) $I_\tau(\lambda, r) \wedge I_\tau(\mu, r) = I_\tau(\lambda \wedge \mu, r)$,

(4) $I_\tau(\lambda, r) \leq I_\tau(\mu, s)$ if $r \geq s$,

(5) $I_\tau(I_\tau(\lambda, r), r) = I_\tau(\lambda, r)$,

(6) $I_\tau(\underline{1} - \lambda, r) = \underline{1} - C_\tau(\lambda, r)$ and $C_\tau(\underline{1} - \lambda, r) = \underline{1} - I_\tau(\lambda, r)$.

Definition 2.6. [13]

A point x of X is called δ -cluster point of λ if $I_\tau(C_\tau(U, r), r) \wedge \lambda = \phi$, for everyopen set U of X containing x. The set of all δ -cluster point of λ is called δ -closure of λ and is denoted $\delta C_\tau(\lambda)$.

Definition 2.6.

A set λ is δ -closed if and only if $\lambda = \delta C_\tau(\lambda, r)$. The complement of a δ -closed set is said to be δ -open [13]. Then δ -interior of a subset λ of X is the union of all δ -open sets of X contained in λ .

Definition 2.8.

Let (X, τ) be a sfts. Then for each $\lambda \in I^X$, $r \in I_0$, λ is called

1. r-fuzzy preopen (resp. r-fuzzy preclosed) [15] set if $\lambda \leq I_\tau(C_\tau(\lambda, r), r)$,

   $(C_\tau(I_\tau(\lambda, r), r) \leq \lambda)$,

2. r-fuzzy α -open (resp. r-fuzzy α -closed) [15] set if $\lambda \leq I_\tau(C_\tau(I_\tau(\lambda, r), r), r)$,

   $(C_\tau(I_\tau(C_\tau(\lambda, r), r), r) \leq \lambda)$,

3. r-fuzzy β -open (resp. r-fuzzy β -closed) [15] set if $\lambda \leq C_\tau(I_\tau(C_\tau(\lambda, r), r), r)$

   $(I_\tau(C_\tau(C_\tau(\lambda, r), r), r) \leq \lambda)$,

4. r-fuzzy θ -open (resp. r-fuzzy θ -closed) [5] set if $\lambda = \theta I_\tau(\lambda, r)$,

   $(\lambda \leq \theta C_\tau(\lambda, r))$, The family of all r-fuzzy preopen (resp. r-fuzzy α-open, r-fuzzy β-open, r-fuzzy θ-open) is denoted by $PO(X)$ (resp. $\alpha O(X)$, $eO(X)$, $\beta O(X)$, $\theta O(X)$.

Lemma: 2.1 [15]

Let λ, μ be two subsets of (X, τ) Then:

(1) λ is r-fuzzy δ-open if and only if $\lambda = I_{\tau}(\lambda, r)$,

(2) $X - \delta I_{\tau}(\lambda, r) = \delta C_{\tau}(X \setminus (\lambda, r))$ and $\delta I_{\tau}(X \setminus (\lambda, r)) = X - \delta I_{\tau}(\lambda, r)$,

(3) $I_{\tau}(\lambda, r) \leq \delta C_{\tau}(\lambda, r)$ (resp. $\delta I_{\tau}(\lambda, r) \leq I_{\tau}(\lambda, r)$)), for any subset λ of X,

(4) $\delta\text{-}C_{\tau}(\lambda \vee \mu, r) = \delta\text{-}C_{\tau}(\lambda, r) \vee \delta\text{-}C_{\tau}(\mu, r)$,
$\delta\text{-}I_{\tau}(\lambda \vee \mu, r) = \delta\text{-}I_{\tau}(\lambda, r) \vee \delta\text{-}I_{\tau}(\mu, r)$,

Definition

A space X is called r-fuzzy extremally disconnected (E.D) [19] if the closure of each r-fuzzy open set in X is open. A space X is called r-fuzzy β-connected [1, 14] if X cannot be expressed as the union of two nonempty disjoint r-fuzzy β-open sets.

Definition

A function $f : (X, \tau) \to (Y, \sigma)$ is called:

(i) r-fuzzy β-continuous [13] if for each r-fuzzy open subset μ of Y, $f^{-1}(\mu) \in \beta O(X, \tau)$..

(ii) r-fuzzy strongly continuous [3, 8], if for every fuzzy subset λ of X, $f(C_{\tau}(\lambda)) \leq f(\lambda)$..

(iii) r-fuzzy weakly open [15] if $f(\eta) \leq I_{\tau}(f(C_{\tau}(\eta, r), r))$ for each r-fuzzy open subset η of X.

(iv) r-fuzzy weakly closed [15] if $C_{\tau}(f(I_{\tau}(F, r), r)) \leq f(F)$ for each r-fuzzy closed subset F of X.

(v) r-fuzzy relatively weakly open [15] provide that $f(\eta)$ is r-fuzzy open in $f(C_{\tau}(\eta, r))$ for every r-fuzzy open subset η of X.

(vi) r-fuzzy almost open, written as (a.o.S) [13] if the image of each r-fuzzy regular open subset η of X is r-fuzzy open set in Y.

(vii) r-fuzzy preopen [13] (resp. r-fuzzy β-open, r-fuzzy α-open) if for each r-fuzzy open subset η of X, $f(\eta)$ is r-fuzzy preopen (resp. $f(\eta)$ is r-fuzzy β-open, f(U) is r-fuzzy α-open) set in Y.

(viii) r-fuzzy preclosed [18] (resp. r-fuzzy β-closed , r-fuzzy α-closed) if for each r-fuzzy closed subset F of X, $f(F)$ is r-fuzzy preclosed (resp. $f(F)$ is r-fuzzy β-closed, $f(F)$ is r-fuzzy α-closed) set in Y.

(ix) r-fuzzy contra-open [13] (resp. r-fuzzy contra-closed [1], r-fuzzy contra β-closed) if $f(\eta)$ is r-fuzzy closed (resp. r-fuzzy open, r-fuzzy β-open) in Y for each r-fuzzy open (resp. r-fuzzy closed, r-fuzzy closed) r-fuzzy subset η of X.

2. Weakly r-fuzzy β-open functions.

In this section, we define the concept of r-fuzzy weak β-openness as a natural dual to the r-fuzzy weak β-continuity due to and Noiri [14] and we obtain several fundamental properties of this new function.

Definition 2.1.

A function $f:(X,\tau)\to(Y,\sigma)$ is said to be r-fuzzy weakly β-open if

$f(\eta)\le\beta(I_\tau(f(C_\tau(\eta,r),r)))$ for each r-fuzzy open set η of X.

Clearly, every r-fuzzy weakly open function is r-fuzzy weakly β-open and every r-fuzzy β-open function is also r-fuzzy weakly β-open, but the converse is not generally true. For,

Example 2.2.

Let $X=Y=\{a,b,c\}$ and $\mu_1,\mu_2\in I^X$ $\mu_2,\mu_3\in I^Y$, $\mu_5\in I^Z$ defined as follows:

$\mu_1(a)=0.4,\mu_1(b)=0.5,\mu_1(c)=0.6,$

$\mu_2(a)=0.4,\mu_2(b)=0.5,\mu_2(c)=0.4,$

$\mu_3(a) = 0.6, \mu_3(b) = 0.5, \mu_3(c) = 0.4,$

$\mu_4(a) = 0.4, \mu_4(b) = 0.5, \mu_4(c) = 0.6,\ \mu_5(a) = 0.6, \mu_5(b) = 0.5, \mu_5(c) = 0.4,$

Define fuzzy topology $\tau, \sigma, \eta = I^X \to I$ as follows

$$\tau(\lambda) = \begin{Bmatrix} 1 \text{ if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} \text{ if } \lambda = \mu_1, \\ \frac{1}{2} \text{ if } \lambda = \mu_2, \\ 0 \text{ if } \cdot \text{ otherwise} \end{Bmatrix}$$

$$\sigma(\lambda) = \begin{Bmatrix} 1 \text{ if } \lambda \in \{\underline{0}, \underline{1}\}, , \\ \frac{1}{2} \text{ if } \lambda = \mu_3, \\ \frac{1}{2} \text{ if } \lambda = \mu_4, \\ \frac{1}{2} \text{ if } \lambda = \mu_3 \vee \mu_4, \\ \frac{1}{2} \text{ if } \lambda = \mu_3 \wedge \mu_4, \\ 0 \text{ if } \cdot \text{ otherwise} \end{Bmatrix}$$

$$\eta(\lambda) = \begin{Bmatrix} 1 \text{ if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} \text{ if } \lambda = \mu_5, \\ 0 \text{ if } \cdot \text{ otherwise} \end{Bmatrix}$$

(i) Then the identity mapping $id_X : (X, \tau) \to (Y, \sigma)$ is $\frac{1}{2}$- fuzzy weakly β-open set but not $\frac{1}{2}$-fuzzy weakly open.

(ii) Then the identity mapping $id_X : (X, \tau) \to (Z, \eta)$ is $\frac{1}{2}$ fuzzy weakly β-open but not $\frac{1}{2}$-fuzzy β-open

Theorem 2.3.

Let X be a r-fuzzy regular space. Then $f:(X,\tau)\to(Y,\sigma)$ is r-fuzzy weakly β-open if and only if f is r-fuzzy β-open.

Proof.

The sufficiency is clear.

For the necessity, let χ be a nonempty r-fuzzy open subset of X. For each x in χ, let η_X be an r-fuzzy open set such that $x\in\eta_X \le C_\tau(\eta_X,r)\le\chi$. Hence we obtain that $\chi=\vee\{\eta_X : x\in\chi\}=\vee\{C_\tau(\eta_X,r):x\in\chi\}$ and,

$$f(\chi)=\vee\{f(\eta_X):x\in\chi\}\le\vee\{\beta I_\tau(f(C_\tau(\eta_X,r)):x\in\chi\}\le\beta_\tau(f(\vee\{C_\tau(\eta_X,r):x\in\chi\})$$
$$=\beta I_\tau(f(\chi,r)).$$

Thus f is r-fuzzy β-open.

Theorem 2.4.

For a function $f:(X,\tau)\to(Y,\sigma)$, the following conditions are equivalent :

(i) f is r-fuzzy weakly β-open.

(ii) $f(\theta I_\tau(\lambda,r))\le\beta I_\tau(f(\lambda,r))$ for every r-fuzzy subset λ of X.

(iii) $\theta I_\tau(f^{-1}(\mu,r))\le f^{-1}(\beta I_\tau(\mu,r))$ for every r-fuzzy subset μ of Y.

(iv) $f^{-1}(\beta C_\tau(\mu,r))\le\theta C_\tau(f^{-1}(\mu,r))$ for every r-fuzzy subset μ of Y.

(v) For each $x\in X$ and each r-fuzzy open subset ϑ of X containing x, there exists a r-fuzzy β-open set η containing $f(x)$ such that $\eta\le f(C_\tau(\vartheta,r))$.

(vi) $f(I_\tau(\omega,r))\le\beta I_\tau(f(\omega,r))$ for each r-fuzzy closed subset ω of X.

(vii) $f(\vartheta)\le\beta I_\tau(f(C_\tau(\vartheta,r),r))$ for each r-fuzzy open subset ϑ of X.

(viii) $f(I_\tau(C_\tau(\vartheta,r),r))\le\beta I_\tau(f(C_\tau(\vartheta,r),r))$ for each r-fuzzy preopen subset ϑ of X.

(X) If $f(\vartheta)\le\beta I_\tau(f(C_\tau(\vartheta,r),r))$ For each r-fuzzy α-open subset ϑ of X.

Proof. (i) $\rightarrow$ (ii) : Let λ be any r-fuzzy subset of X and $x \in \theta I_\tau(\lambda, r)$. Then, there exists an r-fuzzy open set ϑ such that $x \in \vartheta \le C_\tau(\vartheta) \le \lambda$. Then, $f(x) \in f(\vartheta) \le f(C_\tau(\vartheta, r)) \le f(\lambda)$. Since f is r-fuzzy weakly β-open,

$f(\vartheta) \le \beta I_\tau(f(C_\tau(\vartheta, r), r)) \le \beta I_\tau(f(\lambda, r))$. It implies that $f(x) \le \beta I_\tau(f(\lambda, r))$.

This shows that $x \in f^{-1}(\beta I_\tau(f(\lambda, r)))$. Thus $\theta I_\tau(\lambda, r) \le f^{-1}(\beta I_\tau(f(\lambda, r)))$, and so, $f(\theta I_\tau(\lambda, r)) \le \beta I_\tau(f(\lambda, r))$.

(ii) $\rightarrow$ (i) : Let ϑ be an r-fuzzy open set in X. As $\vartheta \le \theta I_\tau(C_\tau(\vartheta, r))$ implies, $f(\vartheta) \le f(\theta I_\tau(C_\tau(\vartheta, r), r)) \le \beta I_\tau(f(C_\tau(\vartheta, r), r))$. Hence f is r-fuzzy weakly β-open.

(ii) $\rightarrow$ (iii) : Let μ be any r-fuzzy subset of Y.

Then by (ii), $f(\theta I_\tau(f^{-1}(C_\tau(\mu, r), r))) \le \beta I_\tau(\vartheta, r)$.

Therefore $\theta I_\tau(f^{-1}(\mu, r)) \le f^{-1}(\beta I_\tau(\mu, r))$.

(iii) $\rightarrow$ (ii) : This is obvious.

(iii) $\rightarrow$ (iv) : Let μ be any r-fuzzy subset of Y. Using (iii), we have

$X - \theta C_\tau(f^{-1}(\mu, r)) = \theta I_\tau(X - f^{-1}(\mu, r)) = \theta I_\tau(f^{-1}(Y - \mu, r))$
$\le f^{-1}(\beta I_\tau(Y - \mu, r)) = f^{-1}(Y - \beta C_\tau(\mu, r)) = X - (f^{-1}(\beta C_\tau(\mu, r))$.

Therefore, we obtain $f^{-1}(\beta C_\tau(\mu, r)) \le \theta C_\tau(f^{-1}(\mu, r))$.

(iv) $\rightarrow$ (iii) : Similarly we obtain, $X - f^{-1}(\beta I_\tau(\mu, r)) \le X - \theta I_\tau(f^{-1}(\mu, r))$,

for every r-fuzzy subset μ of Y, i.e., $\theta I_\tau(f^{-1}(\mu, r)) \le f^{-1}(\beta I_\tau(\mu, r))$.
. (i) $\rightarrow$ (v) : Let $x \in X$ and ϑ be an r-fuzzy open set in X with $x \in X$. Since f is r-fuzzy weakly β-open. $f(x) \in f(\vartheta) \le \beta I_\tau(f(C_\tau(\vartheta, r), r)$. Let $\eta = \beta I_\tau(f(C_\tau(\vartheta, r), r)$. Hence $\eta \le f(C_\tau(\vartheta, r))$., with η containing $f(x)$.

(v) $\rightarrow$ (i) : Let ϑ be an r-fuzzy open set in X and let $y \in f(\vartheta)$. It following from (v) $\eta \le f(C_\tau(\vartheta, r))$ for some η is r-fuzzy β-open in Y containing y. Hence we have, $y \in \eta \le \beta I_\tau(f(C_\tau(\vartheta, r), r))$. This shows that $f(\vartheta) \le \beta I_\tau(f(C_\tau(\vartheta, r), r))$, i.e., f is a r-fuzzy weakly β-open function.

(i) → (vi) → (vii) → (viii) → (ix) → (i) : This is obvious.

Theorem 2.5.

Let $f:(X,\tau)\to(Y,\sigma)$ be a bijective function. Then the following statements are equivalent.

(i) f is r-fuzzy weakly β-open.

(ii) $\beta C_{\tau}(f(\vartheta,r))\le f(C_{\tau}(\vartheta,r))$ for each ϑ is r-fuzzy open of X.

(iii) $\beta C_{\tau}(f(I_{\tau}(\omega,r),r))\le f(\omega)$ for each ω is r-fuzzy closed in X.

Proof. (i) → (iii) : Let ω be a r-fuzzy closed set in X. Then we have $f(X-\omega)=Y-f(\omega)\le\beta I_{\tau}(f(C_{\tau}(X-\omega,r),r))$ and so $Y-f(\omega)\le Y-\beta C_{\tau}(f(I_{\tau}(\omega,r),r))$. Hence $\beta C_{\tau}(f(I_{\tau}(\omega,r),r))\le f(\omega)$.

(iii) → (ii) : Let ϑ be a r-fuzzy open set in X. Since $C_{\tau}(\vartheta,r)$ is a r-fuzzy closed set and $\vartheta\le I_{\tau}(C_{\tau}(\vartheta,r),r)$ by (iii) we have $\beta C_{\tau}(f(\vartheta,r))\le\beta C_{\tau}(f(I_{\tau}(C_{\tau}(\vartheta,r),r),r))\le f(C_{\tau}(\vartheta,r))$.

(ii) → (iii) : Similar to (iii) → (ii) .

(iii) → (i) : Clear.

Theorem 2.6.

If $f:(X,\tau)\to(Y,\sigma)$ is r-fuzzy weakly β-open and r-fuzzy strongly continuous, then f is r-fuzzy β-open.

Proof. Let ϑ be an r-fuzzy open subset of X. Since f is r-fuzzy weakly β-open $f(\vartheta)\le\beta I_{\tau}(f(C_{\tau}(\vartheta,r),r)))$. However, because f is r-fuzzy strongly continuous, $f(\vartheta)\le\beta I_{\tau}(f(\vartheta,r))$ and therefore $f(\vartheta)$ is r-fuzzy β-open.

Example 2.7.

A r-fuzzy β-open function need not be r-fuzzy strongly continuous.

Let X = {a, b, c}, and let τ be the indiscrete topology for X. Then the identity function of (X, τ) onto (X, τ) is a β-open function (hence weakly β-open function) which is not strongly continuous.

Theorem 2.8.

A function $f:(X,\tau)\to(Y,\sigma)$ is r-fuzzy β-open if f is r-fuzzy weakly β-open and relatively r-fuzzy weakly open.

Proof.

Assume f is r-fuzzy weakly β-open and relatively r-fuzzy weakly open. Let ϑ be an r-fuzzy open subset of X and let $y\in f(\vartheta)$. Since f is relatively r-fuzzy weakly open, there is an open subset η of Y for which $f(\vartheta)=f(C_\tau(\vartheta,r))\wedge\eta$. Because f is r-fuzzy weakly β-open, it follows that $f(\vartheta)\le\beta I_\tau(f(C_\tau(\vartheta,r),r))$.Then $y\in\beta I_\tau(f(C_\tau(\vartheta,r),r))\wedge\eta\le f(C_\tau(\vartheta,r))\wedge\eta=f(\vartheta)$ and therefore $f(\vartheta)$ is r-fuzzy β-open.

Theorem 2.9.

If $f:(X,\tau)\to(Y,\sigma)$ is r-fuzzy contra β-closed, then f is a r-fuzzy weakly β open function.

Proof.

Let ϑ be an r-fuzzy open subset of X. Then, we have $f(\vartheta)\le f(C_\tau(\vartheta,r))=\beta I_\tau(f(C_\tau(\vartheta,r),r))$. The converse of Theorem 2.9 does not hold.

Example 2.10.

A r-fuzzy weakly β-open function need not be r-fuzzy contra β-closed is given from Example 2.2(ii). Next, we define a dual form, called complementary r-fuzzy weakly β-open function.

Definition 2.11.

A function $f:(X,\tau)\to(Y,\sigma)$ is called complementary r-fuzzy weakly β--open (written as c.w.β.o) if for each r-fuzzy open set ϑ of X, $f(Fr(\vartheta,r))$ is r-fuzzy β-closed in Y, where $Fr(\vartheta,r)$ denotes the frontier of ϑ.

Example 2.12.

A r-fuzzy weakly β-open need not be r-fuzzy complementary weakly β-open.

Let $X=\{a,b,c\}, Y=\{a,b\}$ and $\mu_1,\mu_2,\mu_3 \in I^X$ $\mu_4,\mu_5,\mu_6 \in I^Y$ defined as follows:

$\mu_1(a)=0.4, \mu_1(b)=0.6, \mu_1(c)=0.6,$

$\mu_2(a)=0.4, \mu_2(b)=0.5, \mu_2(c)=0.6,$

$\mu_3(a)=0.6, \mu_3(b)=0.6, \mu_3(c)=0.6,$

$\mu_4(a)=0.6, \mu_4(b)=0.6, \mu_4(c)=0.5$

$\mu_5(a)=0.5, \mu_5(b)=0.5, \mu_4(c)=0.4$

$\mu_6(a)=0.5, \mu_6(b)=0.4. \mu_4(c)=0.4,$

Define fuzzy topology $\tau, \sigma = I^X \to I$ as follows

$$\tau(\lambda)=\begin{Bmatrix} 1 \; if \; \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} if \; \lambda = \mu_1, \\ \frac{1}{2} if \; \lambda = \mu_2, \\ \frac{1}{2} if \; \lambda = \mu_3, \\ 0 \; if \; \cdot \; otherwise \end{Bmatrix}$$

$$\sigma(\lambda)=\begin{Bmatrix} 1 \; if \; \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} if \; \lambda = \mu_4, \\ \frac{1}{2} if \; \lambda = \mu_5, \\ \frac{1}{2} if \; \lambda = \mu_6, \\ 0 \; if \; \cdot \; otherwise \end{Bmatrix}$$

Then the identity mapping $id_X : (X, \tau) \to (Y, \sigma)$ is $\frac{1}{2}$-fuzzy weakly β-open but not $\frac{1}{2}$-fuzzy complementary weakly β-open.

Example 2.13.

r-fuzzy complementary weakly β-open does not imply r -fuzzy weakly β-open.

Let $X = \{a, b\}$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in I^X$ $\mu_1, \mu_2 \in I^Y$ defined as follows.

$\lambda_1(a) = 0.3, \lambda_1(b) = 0.4, \lambda_1(c) = 0.5,$

$\lambda_2(a) = 0.6, \lambda_2(b) = 0.5, \lambda_2(c) = 0.5,$

$\lambda_3(a) = 0.6, \lambda_3(b) = 0.5, \lambda_3(c) = 0.4,$

$\lambda_4(a) = 0.3, \lambda_4(b) = 0.4, \lambda_4(c) = 0.4,$

$\mu_1(a) = 0.4, \mu_1(b) = 0.7, \mu_1(c) = 0.5,$

$\mu_2(a) = 0.4, \mu_2(b) = 0.1, \mu_2(c) = 0.5.$

Define fuzzy topology $\tau = I^X \to I$ as follows

$$\tau(\lambda) = \begin{cases} 1 \; if \; \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} if \; \lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}, \\ 0 \; if \; \cdot otherwise \quad . \end{cases}$$

$$\sigma(\lambda) = \begin{cases} 1 \; if \; \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} if \; \lambda = \mu_1, \\ \frac{1}{2} if \; \lambda = \mu_2, \\ 0 \; if \; \cdot otherwise \end{cases}$$

Then the identity mapping $id_X : (X,\tau) \to (Y,\sigma)$ is $\frac{1}{2}$-fuzzy weakly β-open but not $\frac{1}{2}$-fuzzy complementary weakly β-open.

Note:

Examples 2.12 and 2.13 demonstrate the independence of complementary $\frac{1}{2}$-fuzzy weakly β-openness and $\frac{1}{2}$-fuzzy weakly β-openness.

Theorem 2.14.

Let $\beta O(X,\tau)$ is r-fuzzy closed under intersections. If $f:(X,\tau)\to(Y,\sigma)$ is bijective r-fuzzy weakly β-open and c.w.β.o, then f is r-fuzzy β-open.

Proof.

Let ϑ be an r-fuzzy open subset in X with $x \in \vartheta$, since f is r-fuzzy weakly β-open, by Theorem 2.3(v) there exists a r-fuzzy β-open set η containing $f(x) = y$ such that $\eta \le f(C_\tau(\vartheta,r))$. Now $Fr(\vartheta,r) = C_\tau(\vartheta,r) - \vartheta$ and thus $x \notin Fr(\vartheta,r)$.. Hence $y \notin Fr(\vartheta,r)$. and therefore $y \in \eta - f(Fr(\vartheta,r))$. Put $\eta_y = \eta - f(Fr(\vartheta,r))$. a r-fuzzy β-open set since f is c.w.β.o. Since $y \in \eta_y, y \in f(Fr(\vartheta,r))$. But $y \notin f(Fr(\vartheta,r))$ and thus $y \notin f(Fr(\vartheta,r)) = f(C_\tau(\vartheta,r)) - f(\vartheta)$ which implies that $y \in f(\vartheta)$. Therefore $f(\vartheta) = \vee\{\eta_y : \eta_y \in \beta O(Y,\sigma), y \in f(\vartheta)\}$. Hence f is r-fuzzy β-open.

Theorem 2.15.

If $f:(X,\tau)\to(Y,\sigma)$ is an a.o.S function, then it is a r-fuzzy weakly β-open function.

Proof.

Let ϑ be an r-fuzzy open set in X. Since f is a.o.S and $I_\tau(C_\tau(\vartheta,r),r))$ is r-fuzzy regular open, $f(I_\tau(C_\tau(\vartheta,r),r))$ is r-fuzzy open in Y and hence $f(\vartheta) \le f(I_\tau(C_\tau(\vartheta,r),r)) \le I_\tau(f(C_\tau(\vartheta,r),r) \le \beta I_\tau(f(C_\tau(\vartheta,r),r))$. This shows that f is r-fuzzy weakly β-open. The converse of Theorem 2.15 is not true in general.

Example 2.16.

A r-fuzzy weakly β-open function need not be r-fuzzy a.o.S.

Let $X=\{a,b,c\}, Y=\{a,b,c\}$ and $\mu_1,\mu_2 \in I^X$ $\nu_1,\nu_2 \in I^Y$ defined as follows:

$\mu_1(a)=0.3, \mu_1(b)=0.2, \mu_1(c)=0.7,$

$\mu_2(a)=0.8, \mu_2(b)=0.8, \mu_2(c)=0.4,$

$\upsilon_1(a)=0.8, \upsilon_1(b)=0.7, \upsilon_1(c)=0.6,$

$\upsilon_2(a)=0.5, \upsilon_2(b)=0.6, \upsilon_2(c)=0.2.$

Define fuzzy topology $\tau = I^X \to I$ as follows

$$\tau(\lambda)=\begin{cases} 1 \ if \ \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2} if \ \lambda = \mu_1, \\ \frac{1}{3} if \ \lambda = \mu_2, \\ \frac{2}{3} if \ \lambda = \mu_1 \vee \mu_2, \\ \frac{2}{3} if \ \lambda = \mu_1 \wedge \mu_2, \\ 0 \ if \ \cdot \ otherwise \end{cases}$$

$$\sigma(\lambda)=\begin{cases} 1 \ if \ \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2} if \ \lambda = \nu_1, \\ \frac{1}{2} if \ \lambda = \nu_2, \\ 0 \ if \ \cdot \ otherwise \end{cases}$$

Then the identity mapping $id_X:(X,\tau)\to(Y,\sigma)$ is $\frac{1}{2}$-fuzzy weakly β-open but not $\frac{1}{2}$-fuzzy a.o.S, since $I_\tau(f(I_\tau(C_\tau(\lambda,\frac{1}{2}),\frac{1}{2}),\frac{1}{2})=\phi.$

Lemma 2.17.

If $f:(X,\tau)\to(Y,\sigma)$ is a r-fuzzy continuous function, then for any r-fuzzy subset ϑ of X, $f(C_\tau(\vartheta,r))\leq C_\tau(f(\vartheta,r))$.

Theorem 2.18. If $f:(X,\tau)\to(Y,\sigma)$ is a r-fuzzy weakly β-open and r-fuzzy continuous function, then f is a r-fuzzy β-open function.

Proof.

Let ϑ be a r-fuzzy open set in X. Then by r-fuzzy weak β-openness of f, $f(\vartheta)\leq \beta I_\tau(f(C_\tau(\vartheta,r),r))$. Since f is r-fuzzy continuous $f(C_\tau(\vartheta,r))\leq C_\tau(f(\vartheta,r))$. Hence we obtain that, $f(\vartheta)\leq \beta I_\tau(f(C_\tau(\vartheta,r),r))\leq \beta I_\tau(C_\tau(f(\vartheta,r),r))\leq C_\tau(I_\tau(C_\tau(f(\vartheta,r),r),r))$. Therefore, $f(\vartheta)\leq C_\tau(I_\tau(f(\vartheta,r)))$ which shows that $f(\vartheta)$ is a r-fuzzy β-open set in Y. Thus, f is a r-fuzzy β-open function.

Since every r-fuzzy strongly continuous function is r-fuzzy continuous we have the following corollary.

Corollary 2.19.

If $f:(X,\tau)\to(Y,\sigma)$ is an injective r-fuzzy weakly β-open and r-fuzzy strongly continuous function. Then f is r-fuzzy β-open.

Theorem 2.20.

If $f:(X,\tau)\to(Y,\sigma)$ is a injective r-fuzzy weakly β-open function of a space X onto a r-fuzzy β–connected space Y, then X is r-fuzzy connected.

Proof.

Let us assume that X is not r-fuzzy connected. Then there exist nonempty r-fuzzy open sets ϑ_1 and ϑ_2 such that $\vartheta_1\wedge\vartheta_2=\phi$ and $\vartheta_1\vee\vartheta_2=X$. Hence we have $f(\vartheta_1)\wedge f(\vartheta_2)=\phi$ and $f(\vartheta_1)\vee f(\vartheta_2)=Y$. Since f is r-fuzzy weakly β-open, we have $f(\vartheta_i)\leq \beta I_\tau(f(C_\tau(\vartheta_i,r),r)$ for $i=1,2$ and since ϑ_i is r-fuzzy open and also r-fuzzy closed, we have $f(C_\tau(\vartheta_i,r))=f(\vartheta_i)$ for $i=1,2$. Hence $f(\vartheta_i)$ is r-fuzzy β-open in Y for $i=1,2$. Thus, Y has been decomposed into two non-empty disjoint r-fuzzy β-open sets . This is contrary to the hypothesis that Y is a r-fuzzy β–connected space. Thus X is connected.

Recall, that a space X is said to be hyperconnected [12, 13] if every nonempty open subset of X is dense in X.

Theorem 2.21.

If X is a r-fuzzy hyperconnected space, then a function $f:(X,\tau)\to(Y,\sigma)$ is r-fuzzy weakly β-open if and only if $f(X)$ is r-fuzzy β-open in Y.

Proof.

The sufficiency is clear. For the necessity observe that for any r-fuzzy open subset ϑ of X, $f(\vartheta)\le f(X)=\beta I_\tau(f(X,r))=\beta I_\tau(f(C_\tau(\vartheta,r),r))$.

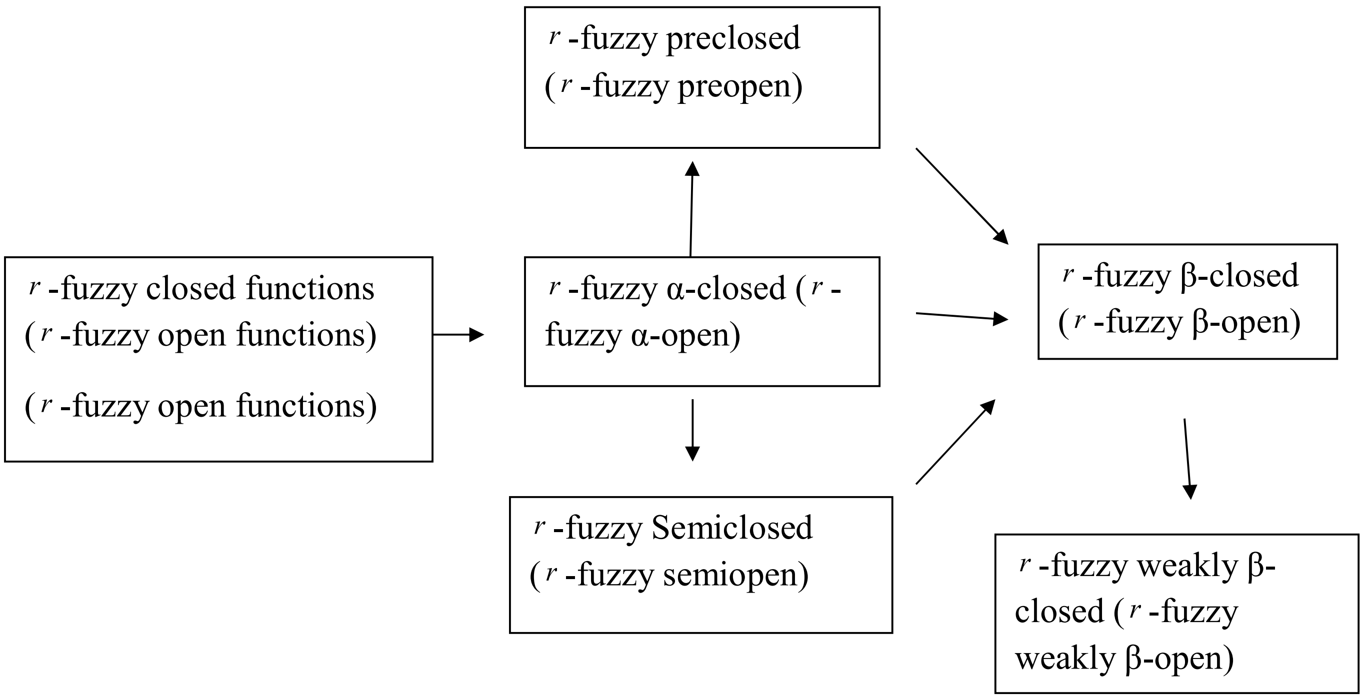
3. r-fuzzy Weakly β-closed functions.

Now, we define the generalized form of r-fuzzy β-closed functions

Definition 3.1.

A function $f:(X,\tau)\to(Y,\sigma)$ is said to be r-fuzzy weakly β-closed if $\beta C_\tau(f(I_\tau(F,r),r))\le f(F)$ for each closed set F in X.

The implications between r-fuzzy weakly β-closed (res. r-fuzzy weakly β-open) functions and other types of r-fuzzy closed (resp. r-fuzzy open) functions are given by the following diagram.



The converse of these statements are not necessarily true, as shown by the following examples.

Example.3.2

An injective function from a fuzzy discrete space into an fuzzy indiscrete space is r-fuzzy β-open and r-fuzzy β-closed, but neither r-fuzzy α-open nor r-fuzzy α-closed.

Example.3.3

Let $X = \{a, b\}$ and $Y = \{p, q\}$ .Define $\lambda_1, \lambda_2 \in I^X$ $\mu_1, \mu_2 \in I^Y$ as follows:

$\lambda_1(a) = 0.1, \lambda_1(b) = 0.2,$

$\lambda_2(a) = 0.2, \lambda_2(b) = 0.1,$

$\mu_1(p) = 0.6, \mu_1(q) = 0.7,$

$\mu_2(p) = 0.7, \mu_2(q) = 0.6.$

Define fuzzy topology $\tau_1, \tau_2 = I^X \to I$ as follows

$$\tau_1(\lambda) = \begin{cases} 1 \ if \ \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{2}{3} \ if \ \lambda = \lambda_1, \\ \frac{1}{4} \ if \ \lambda = \lambda_2, \\ \frac{1}{4} \ if \ \lambda = \lambda_1 \vee \lambda_2, \\ 0 \ if \ \cdot otherwise \end{cases}$$

$$\tau_2(\mu) = \begin{cases} 1 \ if \ \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{2}{3} \ if \ \mu = \mu_1, \\ \frac{1}{4} \ if \ \mu = \mu_2, \\ \frac{1}{4} \ if \ \mu = \mu_1 \vee \mu_2 \\ 0 \ if \ \cdot otherwise \end{cases}$$

Then the mapping $f:(X,\tau_1)\to(Y,\tau_2)$ define by $f(a)=q, f(b)=q$. Then f is $\frac{2}{3}$-fuzzy α-open and $\frac{2}{3}$-fuzzy α-closed but neither $\frac{2}{3}$-fuzzy open nor $\frac{2}{3}$-fuzzy closed

Example 3.4.

Let f : (X, τ) → (Y, σ) be the function from Example 2.2. Then it is shown that f is r-fuzzy weakly β-closed which is not r-fuzzy weakly closed.

Example 3.5.

Let $X=\{a,b,c\}$ and $Y=\{p,q,r\}$.Define $\lambda_1,\lambda_2 \in I^X$ $\mu_1,\mu_2 \in I^Y$ as follows:

$\lambda_1(a)=0.3, \lambda_1(b)=0.7, \lambda_1(c)=0.7$

$\lambda_2(a)=0.7, \lambda_2(b)=0.3, \lambda_2(c)=0.3,$

$\mu_1(p)=0.1, \mu_1(q)=0.3, \mu_1(r)=0.3$

$\mu_2(p)=0.3, \mu_2(q)=0.1, \mu_2(q)=0.1.$

Define fuzzy topology $\tau_1, \tau_2 = I^X \to I$ as follows

$$\tau_1(\lambda) = \begin{cases} 1 \; if \; \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} \; if \; \lambda = \lambda_1, \\ \frac{1}{3} \; if \; \lambda = \lambda_2, \\ \frac{1}{3} \; if \; \lambda = \lambda_1 \vee \lambda_2, \\ 0 \; if \; \cdot \; otherwise \end{cases}$$

$$\tau_2(\mu) = \begin{cases} 1 \; if \; \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} \; if \; \mu = \mu_1, \\ \frac{1}{3} \; if \; \mu = \mu_2, \\ \frac{1}{3} \; if \; \mu = \mu_1 \vee \mu_2 \\ 0 \; if \; \cdot \; otherwise \end{cases}$$

Then the mapping $f:(X,\tau_1)\to(Y,\tau_2)$ define by $f(a)=q, f(b)=q, f(c)=r.$ Then f is $\frac{1}{2}$-fuzzy β-closed ($\frac{1}{2}$-fuzzy β-open) but not $\frac{1}{2}$-fuzzy semi open ($\frac{1}{2}$-fuzzy semi closed) .

Example 3.6.

Let $X=\{a,b,c\}$ and $Y=\{p,q,r\}$.Define $\lambda_1, \lambda_2 \in I^X$ $\mu_1 \in I^Y$ as follows:

$\lambda_1(a)=0.4, \lambda_1(b)=0.6, \lambda_1(c)=0.4$

$\lambda_2(a)=0.6, \lambda_2(b)=0.4, \lambda_2(c)=0.4,$

$\mu_1(p)=0.6, \mu_1(q)=0.4, \mu_1(r)=0.5$

Define fuzzy topology $\tau_1, \tau_2 = I^X \to I$ as follows

$$\tau_1(\lambda) = \begin{cases} 1 \; if \; \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} \; if \; \lambda = \lambda_1, \\ \frac{1}{2} \; if \; \lambda = \lambda_2, \\ \frac{1}{2} \; if \; \lambda = \lambda_1 \vee \lambda_2, \\ \frac{1}{2} \; if \; \lambda = \lambda_1 \wedge \lambda_2 \\ 0 \; if \; \cdot \; otherwise \end{cases}$$

$$\tau_2(\mu) = \begin{cases} 1 \; if \; \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2} \; if \; \mu = \mu_1, \\ 0 \; if \; \cdot \; otherwise \end{cases}$$

Then the identity mapping $id_X : (X, \tau_1) \to (Y, \tau_2)$ define by $f(a) = a, f(b) = b, f(c) = c$. Then f is $\frac{1}{2}$-fuzzy β-closed ($\frac{1}{2}$-fuzzy β-open) but not $\frac{1}{2}$-fuzzy preclosed ($\frac{1}{2}$-fuzzy preopen) .

Theorem 3.4.

For a function $f : (X, \tau) \to (Y, \sigma)$, the following conditions are equivalent.

(i) f is r-fuzzy weakly β-closed.

(ii) $\beta C_\tau(f(\vartheta)) \le f(C_\tau(\vartheta))$ for every r-fuzzy open set ϑ of X .

(iii) $\beta C_\tau(f(\vartheta)) \le f(C_\tau(\vartheta))$ for each r-fuzzy regular open subset ϑ of X ,

(iv) For each subset F in Y and each r-fuzzy open set F in X with $f^{-1}(F) \le \vartheta$, ,there exists a r-fuzzy β-open set λ in Y with $F \le \lambda$ and $f^{-1}(F) \le C_\tau(\vartheta)$, ,

(v) For each point y in Y and each open set ϑ in X with $f^{-1}(y) \leq \vartheta,$, there exists a r-fuzzy β-open set λ in Y containing y and $f^{-1}(\lambda) \leq \vartheta,$,

(vi) $\beta C_\tau(f(I_\tau(C_\tau(\vartheta,r),r),r) \leq f(C_\tau(\vartheta,r))$ for each set ϑ in X,

(vii) $\beta C_\tau(f(I_\tau(\theta C_\tau(\vartheta,r),r),r) \leq f(\theta C_\tau(\vartheta,r))$ for each set ϑ in X,

(viii) $\beta C_\tau(f(\vartheta,r),r) \leq f(C_\tau(\vartheta,r))$ for each r-fuzzy β-open set ϑ in X.

Proof. $(i) \to (ii)$. Let ϑ be any r-fuzzy open subset of X. Then $\beta C_\tau(f(\vartheta,r)) = \beta C_\tau(f(I_\tau(\vartheta,r),r) \leq \beta C_\tau(f(I_\tau(C_\tau(\vartheta,r),r),r) \leq f(C_\tau(\vartheta,r))$.

$(ii) \to (i)$. Let F be any r-fuzzy closed subset of X. Then,

$\beta C_\tau(f(I_\tau(\vartheta,r),r) \leq f(C_\tau(I_\tau(F,r),r) \leq f(C_\tau(F,r)) = f(F)$.

It is clear that: $(ii) \to (vii)$, $(iv) \to (v)$, and $(i) \to (vi) \to (viii) \to (iii) \to (i)$.

To show that $(iii) \to (iv)$,: Let F be a fuzzy subset in Y and let ϑ be fuzzy open in X with $f^{-1}(F) \leq \vartheta$. Then $f^{-1}(F) \wedge C_\tau(X - C_\tau(\vartheta,r),r) = \phi$. and consequently, $F \wedge f\ (C_\tau(X - C_\tau(\vartheta,r),r) = \phi$.. Since $X - C_\tau(\vartheta,r)$ is r-fuzzy regular open, $F\ \ F \wedge \beta C_\tau(f(X - C_\tau(\vartheta,r),r) = \phi$ by (iii).

Let $\lambda = Y - \beta C_\tau(f(X - C_\tau(\vartheta,r),r)$.. Then λ is r-fuzzy β-open with $F \leq \lambda$ and $f^{-1}(\lambda) \leq X - f^{-1}(\beta C_\tau(f(X - C(\vartheta,r),r)) \leq X - f^{-1}f(X - C_\tau(\vartheta,r)) \leq C_\tau(\vartheta,r)$.

$(vii) \to (i)$: It is suffices see that $\theta C_\tau(\vartheta,r) = C_\tau(\vartheta,r)$ for every open sets ϑ in X.

$(v) \to (i)$: Let F be closed in X and let $y \in Y - f(F)$. Since $f^{-1}(y) \leq X - F$, there exists a r-fuzzy β-open λ in Y with $y \in \lambda$ and $f^{-1}(\lambda) \leq C_\tau(X - F,r) = X - I_\tau(F,r)$ by (v). Therefore $\lambda \wedge f(I_\tau(F,r) = \phi$, so that $y \in Y - \beta C_\tau(f(I_\tau(F,r),r)$.. Thus $(v) \to (i)$. Finally, for

$(vii) \to (viii)$: Note that $\theta C_\tau(\vartheta,r) = C_\tau(\vartheta,r)$ for each r-fuzzy β-open subset ϑ in X. The following theorem the proof is mostly straightforward and is omitted

Theorem 3.5.

For a function $f:(X,\tau)\to(Y,\sigma)$ the following conditions are equivalent :

(i) f is r-fuzzy weakly β-closed,

(ii) $\beta C_\tau(f(I_\tau(F,r),r)\leq f(F)$ for each r-fuzzy β-closed subset F in X,

(iii) $\beta C_\tau(f(I_\tau(F,r),r)\leq f(F)$ for every r-fuzzy α-closed subset F in X.

Remark 3.6.

By Theorem 2.5, if $f:(X,\tau)\to(Y,\sigma)$ is a bijective function, then f is r-fuzzy weakly β-open if and only if f is r-fuzzy weakly β-closed. Next we investigate conditions under which r-fuzzy weakly β-closed functions are r-fuzzy β-closed.

Theorem 3.7.

If $f:(X,\tau)\to(Y,\sigma)$ is r-fuzzy weakly β-closed and if for each fuzzy closed subset F of X and each fiber $f^{-1}(y)\leq X-F$ there exists a open ϑ of X such that $f^{-1}(y)\leq\vartheta\leq C_\tau(\vartheta,r)\leq X-F$. Then f is r-fuzzy β-closed.

Proof. Let F is any closed subset of X and let $y\in Y-f(F)$. Then $f^{-1}(y)\wedge F=\phi$ and hence $f^{-1}(y)\leq X-F$.. By hypothesis, there exists a open ϑ of X such that $f^{-1}(y)\leq\vartheta\leq C_\tau(\vartheta,r)=X-F$.. Since f is r-fuzzy β-weakly -closed by Theorem 3.4, there exists a r-fuzzy β-open η in Y with $y\in\eta$ and $f^{-1}(\eta)\leq C_\tau(\vartheta,r)$. Therefore, we obtain $f^{-1}(\eta)\wedge F=\phi$ and hence $\eta\wedge f(F)=\phi$, this shows that $y\notin\beta C_\tau f(F,r)$. Therefore, $f(F)$ is r-fuzzy β-closed in Y and f is r-fuzzy β-closed.

Theorem 3.8.

If $f:(X,\tau)\to(Y,\sigma)$ is contra-open, then f is weakly β-closed.

Proof. Let F be a closed subset of X. Then,
$\beta C_\tau f(I_\tau(F,r),r)\leq f(I_\tau(F,r))\leq f(F)$.

Theorem 3.9.

If $f:(X,\tau)\to(Y,\sigma)$ is one-one and r-fuzzy weakly β-closed, then for every subset F of Y and every open set ϑ in X with $f^{-1}(F)\le\vartheta$, there exists a r-fuzzy β-closed set μ in Y such that $F\le\mu$ and $f^{-1}(\mu)\le C_\tau(\vartheta,r)$.

Proof.

Let F be a subset of Y and let ϑ be a open subset of X with $f^{-1}(F)\le\vartheta$. Put $\mu=\beta C_\tau(f(I_\tau(C_\tau(\vartheta,r),r),r)$, then μ is a r-fuzzy β-closed subset of Y such that $F\le\mu$ since
$F\le f(\vartheta)\le f(I_\tau(C_\tau(\vartheta,r),r)\le\beta C_\tau(f(I_\tau(C_\tau(\vartheta,r),r),r)=\mu$. And since f is r-fuzzy weakly β-closed, $f^{-1}(\mu)\le C_\tau(\vartheta,r)$.

Taking the set F in Theorem 3.9 to be y for $y\in Y$ we obtain the following result.

Corollary 3.10.

If $f:(X,\tau)\to(Y,\sigma)$ is one-one and r-fuzzy weakly β-closed, then for every point y in Y and every open set ϑ in X with $f^{-1}(y)\le\vartheta$, there exists a r-fuzzy β-closed set μ in Y containing y such that $f^{-1}(\mu)\le C_\tau(\vartheta,r)$.
Recall that, a set F in a space X is r-fuzzy θ-compact if for each cover Ω of F by open ϑ in X, there is a finite family $\vartheta_1,\vartheta_2.\vartheta_3,...,\vartheta_n$ in Ω such that $F\le I_\tau(\vee\{C_\tau(\vartheta_i\}:i=1,2,...,n\})$ [16].

Theorem 3.11.

If $f:(X,\tau)\to(Y,\sigma)$ is r-fuzzy weakly β-closed with all fibers θ-closed, then $f(F)$ is r-fuzzy β-closed for each θ-compact F in X.

Proof.

Let F be r-fuzzy θ-compact and let $y\in Y-f(F)$. Then $f^{-1}(y)\wedge F=\phi$ and for each $x\in F$ there is an open U_x containing x in X and $C_\tau(\vartheta_x)\wedge f^{-1}(y)=\phi$.. Clearly $\Omega=\{U_x:x\in F\}$ is an open cover of F and since F is r-fuzzy θ-compact, there is a finite family $\{\vartheta_{x_1},\vartheta_{x_2}.\vartheta_{x_3},...,\vartheta_{x_n}\}$ in Ω such that $F\le I_\tau(\lambda,r)$, where $\lambda=\vee\{C_\tau(\vartheta_{x_i}):i=1,2,...,n\}$ Since f is r-fuzzy weakly β-closed by Theorem

2.5 there exists a r-fuzzy β-open μ in Y with $f^{-1}(y) \leq f^{-1}(\mu) \leq C_\tau(X-\lambda)X - I_\tau(\lambda, r) \leq X - F$.. Therefore $y \in \mu$ and $\mu \wedge f(F) = \phi$. Thus $y \in Y - \beta C_\tau(f(F, r))$. This shows that $f(F)$ is r-fuzzy β-closed. Two non empty subsets λ and μ in X are strongly separated [16], if there exist open sets ϑ and η in X with $\lambda \leq \vartheta$ and $\mu \leq \eta$ and $C_\tau(\vartheta, r) \leq C_\tau(\eta, r) = \phi$. If λ and μ are singleton sets we may speak of points being strongly separated. We will use the fact that in a r-fuzzy normal space, disjoint closed sets are strongly separated. Recall that a space X is said to be r-fuzzy β-Hausdorff or in short $\beta - T_2$ [10], if for every pair of distinct points x and y, there exist two r-fuzzy β-open sets ϑ and η such that $x \in \vartheta$ and $y \in \eta$ and $\vartheta \wedge \eta = \phi$.

Theorem 3.12.

If $f:(X,\tau) \to (Y,\sigma)$ is a r-fuzzy weakly β-closed surjection and all pairs of disjoint r-fuzzy fibers are strongly separated, then Y is $\beta - T_2$.

Proof.

Let y and z be two points in Y. Let ϑ and η be open sets in X such that $f^{-1}(y) \in \vartheta$ and $f^{-1}(z) \in \eta$ respectively with $C_\tau(\vartheta) \wedge C_\tau(\eta) = \phi$. By r-fuzzy β-closedness (Theorem 3.4(v)) there are r-fuzzy β-open sets F and μ in Y such that $y \in F$ and $z \in \mu$, $f^{-1}(F) \leq C_\tau(\vartheta)$ and $f^{-1}(\mu) \leq C_\tau(\eta)$. Therefore $F \wedge \mu = \phi$, because $C_\tau(\vartheta) \wedge C_\tau(\eta) = \phi$ and f surjective. Then Y is $\beta - T_2$.

Corollary 3.13.

If $f:(X,\tau) \to (Y,\sigma)$ is r-fuzzy weakly β-closed surjection with all fibers closed and X is r-fuzzy normal, then Y is $\beta - T_2$.

Corollary 3.14.

If $f:(X,\tau) \to (Y,\sigma)$ is continuous r-fuzzy weakly β-closed surjection with X is a r-fuzzy compact T_2 space and Y is a r-fuzzy T_1 space, then Y is r-fuzzy compact $\beta - T_2$ space.

Proof.

Since f is a r-fuzzy continuous surjection and Y is a r-fuzzy T_1 space, Y is r-fuzzy compact and all fibers are r-fuzzy closed. Since X is r-fuzzy normal Y is also $\beta - T_2$.

Definition 3.15.

A topological space X is said to be r-fuzzy quasi H-closed (resp. r-fuzzy β-space), if every r-fuzzy open (resp. r-fuzzy β-closed) cover of X has a finite subfamily whose closures cover X. A subset λ of a r-fuzzy topological space X is r-fuzzy quasi H-closed relative to X (resp. r-fuzzy β-space relative to X) if every cover of λ by r-fuzzy open (resp. r-fuzzy β-closed) sets of X has a finite subfamily whose r-fuzzy closures cover λ.

Lemma 3.16.

A function $f:(X,\tau)\to(Y,\sigma)$ is open if and only if for each $\mu \le Y$, $f^{-1}(C_\tau(\mu,r)) \le C_\tau(f^{-1}(\mu,r))$ [9].

Theorem 3.17.

Let X be an extremally disconnected space and $\beta O(X,\tau)$ closed under finite intersections. Let $f:(X,\tau)\to(Y,\sigma)$ be an open weakly β-closed function which is one-one and such that $f^{-1}(y)$ is quasi H-closed relative to X for each y in Y. If G is β-space relative to Y then $f^{-1}(G)$ is quasi H-closed.

Proof.

Let $\{\eta_\alpha : \alpha \in I\}$, (I being the index set) be an open cover of $f^{-1}(G)$. Then for each $y \in G \wedge f(X)$, $f^{-1}(y) \le \vee\{C_\tau(\eta_\alpha, r) : \alpha \in I(y)\} = H_y$ for some finite subfamily $I(y)$ of I. Since X is extremally disconnected each $C_\tau(\eta_\alpha, r)$ is open, hence H_y is open in X. So by Corollary 3.10, there exists a β-closed set ϑ_y containing y such that $f^{-1}(\vartheta_y) \le C_\tau(H_y, r)$. Then,

$\{\eta_y : y \in G \wedge f(X)\} \vee \{Y - f(X)\}$ is a β-closed cover of G,

$G \le \vee\{C_\tau(\vartheta_y, r) : y \in K\} \vee \{C_\tau(Y - f(X)\}$ for some finite subset K of $G \wedge f(X)$.

Hence and by Lemma 3.16,

$$f^{-1}(G) \le \vee\{f^{-1}(C_\tau(\vartheta_y, r) : y \in K\} \vee \{f^{-1}(C_\tau(Y - f(X)\}$$
$$\le \vee\{C_\tau(f^{-1}(\vartheta_y, r) : y \in K\} \vee \{C_\tau(f^{-1}(Y - f(X)))\} \le \{C_\tau(f^{-1}(\vartheta_y, r)) : y \in K\},$$

so $f^{-1}(G) \le \vee\{C_\tau(\eta_\alpha, r) : \alpha \in I(y), y \in K\}$. Therefore $f^{-1}(G)$ is quasi H-closed.

Corollary 3.18.

Let $f : (X, \tau) \to (Y, \sigma)$ be as in Theorem 3.17. If Y is β-space, then X is quasi-H-closed.

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