Non-Abelian Finite Groups OF Semi Direct Products

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ABSTRACT

The purpose of exploring infinite groups in this thesis was to produce non-abelian finite simple groups as homomorphic images. These infinite groups are semi-direct products known as progenitors. The permutation progenitors studied were: $2^{*8} : 2^2 \cdot A_4$, $2^{*10} : D_{20}$, $2^{*4} : C_4$, $2^{*7} : (7 : 6)$, $3^{*3} : S_3$, $2^{*15} : ((5 \times 3) : 2)$, and $2^{*20} : A_5$. When we factored said progenitors by an appropriate number of relations, we produced several original symmetric presentations and constructions of linear groups, other classical groups and sporadic groups. We have found original symmetric presentations of several important groups, including: $PGL_2(7)$, $PSL_2(8)$, $PSL_2(11)$, $PGL_2(11)$, $PGL_2(13)$, $PSL_2(19)$, $PGL_2(29)$, $PSL_2(41)$, $PSL_2(71)$, J_2, $U(3, 4)$, $U(3, 5)$, M_{11}, and M_{22}.

When solving various extension problems, we are able to identify the isomorphism types of the finite images we discovered. We present proofs of the four types of extension problems: Direct Products, Semi-Direct Products, Central Extensions, and Mixed Extensions. We perform manual double coset enumeration with the support of a computer-based program, Magma, to construct Cayley diagrams of the finite groups: $3^2 : S_3$, M_{11}, $PSL_2(19)$, $PSL_2(7)$, S_4, and $U_3(5) : 2$.

Key words

No-abelian group

Semi-Direct products

Permutation

Sporadic Groups

Introduction

Progenitors factored by one or more relations frequently give non-abelian simple groups and even sporadic groups as their homomorphic images. The main goal of this thesis was to obtain original symmetric presentations and constructions of linear groups, other classical groups and sporadic groups by factoring permutation progenitors. Let G be a group and $T = \{t_o, t_1, ..., t_{n-1}\}$ where $T_i =< t_i >$, with $i = 0, 1, ..., n - 1$, is the cyclic subgroup generated by t_i and let N be a subgroup of S_n that acts transitively and faithfully on T called the **control subgroup**.

In the semi-direct product, $2^{*n} : N = \{\pi w | \pi \in N, w$ is a reduced word in $t_i\}$, N acts by conjugation as permutations of the n involutary symmetric generators. Every element of the progenitor can be represented as a word of πw. We want to factor the progenitor by relations of the form $\pi w(t_1, ..., t_n)$, giving us a finite homomorphic image of the infinite progenitor. We will then perform double coset enumeration of some of these finite groups to find the double cosets and determine the number of single cosets each contains. We will use a Cayley Diagram to demonstrate a graphical representation of this process.

Our goal is to factor $2^{*n} : N$ by relations, that equate elements of N to the product of t_is, resulting in finite homomorphic images. Once we find all of the relations, we will perform double coset enumeration of G over N . Hence, we will find all of the double cosets and find the total number of single cosets each double coset contains. We will have completed the double coset enumeration when the set of right cosets obtained is closed under the operation of right multiplication. Thus, we will find the order of G once we find all relations, perform double coset enumeration of G over N , create the Cayley Graph and obtain the index of N in G.

2.1 Permutation Progenitor

In this section we will discuss the technique we used to write presentations for permutation progenitors. We wish to write a presentation for permutation progenitors of the form $2^{*n} : N$. We first choose a control group transitive on n letters, denoted as N . Once we have a presentation for our control group N , we introduce a symmetric generator typically labeled as t, where t is a generator of a free product group. Throughout this thesis we will label our t as t_1, with the exception of one example. This example will clearly state the labeling for t. In order to give our t the name t_1, we must find the generator of our point stabilizing group, hence N^1. Allowing our t to commute with such a generator ensures our t is t_1. In general, the progenitors $\mathbf{(2)}^{*n} : N$ will take on the following form:

$$< generators\ of\ N, t \mid presentation\ of\ N, t^{\mathbf{2}}, (t, N^1) > .$$

In the following example, we will be demonstrating the process of writing a presentation for a permutation progenitor.

Example 2.2.1. ***Writing a Presentation for the Permutation Progenitor,*** $2^{*10} : D_{20}$

We wish to write a presentation for a progenitor of the form $2^{*10} : N$. The 2 in the free product 2^{*10} represents the order of our $t_i s$. The 10 represents the amount of $t_i s$ we have of order 2. Then control group N must be transitive on 10 letters. With the help of Magma's database of stored groups, we find our desired transitive group, D_{20}. We will be using Magma to assist with finding the presentation for this control group N.

As stated previously we must include a presentation of the control group in our progenitor presentation. Since we have 10 $t_i s$, it's convenient to select the Symmetric Group 10, a set of size 10, to begin this process. We let our control group be a subgroup of S_{10} generated by the permutations Magma provided us with: a, b and c. We store this information into Magma as follows.

```
S:=Sym(10);
A:=S!(1, 3, 9, 7, 8)(2, 6, 4, 10, 5);
B:=S!(1, 2)(3, 5)(4, 7)(6, 8)(9, 10);
C:=S!(1, 4)(2, 7)(3, 10)(5, 9)(6, 8);
N:=sub<S|A,B,C>;#N
```

We discover that the order of N is 20. Now we will construct an FP-Group of N to give a presentation in terms of the generators a, b, and c. (CBFS13) We find the FP-Group with the following command.

```
FPGroup(N);
> Finitely presented group on 3 generators
```

```
Relations
$.2^2 = Id($)
$.3^2 = Id($)
($.1^-1 * $.2)^2 = Id($)
$.1^-1 * $.3 * $.1 * $.3 = Id($)
($.2 * $.3)^2 = Id($)
$.1^-5 = Id($)
```

This presents us with relations of the finitely presented group on three generators. The relations given, $.1, $.2, and $.3 are labeled as *a, b*, and *c*, respectively. The construction of the presentation for N is now completed:

$$N =< a, b, c | b^2, c^2, (a^{-1}b)^2, a^{-1}cac, (bc)^2, a^{-5} >.$$

The presentation of our control group can now be written at the beginning of the progenitor presentation.

```
G<a,b,c>:=Group<a,b,c|b^2,c^2,(a^-1*b)^2,a^-1*c*a*c,(b*c)^2,a^-5>;
```

One way to verify this group so far is the control group, is to examine the size of the group. Before constructing the FP-Group of N , we discovered the order of N was 20. When we check the size of this group G, we confirm it is indeed 20.

The last step to writing the presentation of the progenitor is to state the order of our $t_i s$ and allow t to commute with a point stabilizer in N . Stabilizing 1, $t \sim t_1$, suggests t commutes with the stabilizer 1 in N. We ask Magma to provide us with a permutation group N^1 acting on a set of cardinality 10.

```
N1:=Stabiliser(N,1);
N1;
```

The output given is a permutation in the stabilizer, $(2, 5)(3, 8)(6, 10)(7, 9)$, which needs to be changed into a word in the generators of the control group. The Schreier System allows us to change permutations into words. Generator a is the only generator with a distinct inverse, thus we use the following Schreier System.

```
Sch:=SchreierSystem(G,sub<G|Id(G)>);
ArrayP:=[Id(N): i  in [1..#N]];
for  i in [2..#N]  do
P:=[Id(N):  l  in  [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=A^-1; end if; end for;
PP:=Id(N); for k in [1..#P] do PP:=PP*P[k]; end for; ArrayP[i]:=PP;
end for; for i in [1..#N] do
if ArrayP[i] eq S!(2, 5)(3, 8)(6, 10)(7, 9)
then Sch[i]; end if; end for;
```

The word in the generators of the control group that allows t to commute with the stabilizer 1 in N, is ba^2c. Adding t, the order of t, and this word that commutes with t to the presentation of N, gives us the completed presentation of the permutation progenitor, $2^{*10} : (D_{20})$. We store our presentation in Magma,

$G =< a, b, c, t | b^2, c^2, (a^{-1}b)^2, a^{-1}cac, (bc)^2, a^{-5}, t^2, (t, ba^2c) >.$

```
G<a,b,c,t>:=Group<a,b,c,t|b^2,c^2,(a^-1*b)^2,a^-1*c*a*c,(b*c)^2,a^-5,
t^2,(t,b*a^2*c)>;
```

Curtis' Famous Lemma gives us light on how to find relations in hopes of finding our desired sporadic groups. Two other methods in finding relations are also discussed in the following sec

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