# SPLIT-NEIGHBOURHOOD OF A GRAPH

Shashidhar G. Kumbar

**Assistant Professor**

**Department of Mathematics**

**Govt. First Grade College, Muddebihal**

**Karnataka**

## ABSTRACT

A Neighbourhood set S of a graph G is a Split Neighbourhood set if the induced subgraph $<V - S>$ is disconnected. The split-Neighbourhood number $n_s(G)$ is the minimum cardinality of a split-Neighbourhood set. In this paper, we have obtained bounds for $n_s(G)$in terms of order, size and other parameters of graphs.

**Keywords:** Domination number, Split Domination Number, Split-Neighbourhood.

**Mathematics Subject Classification:** 05C69.

## 1. INTRODUCTION

The graphs considered here are finite, undirected, without loops or multiple edges and connected. Unless otherwise stated, all graphs are assumed to have '$p$' vertices and '$q$' edges.

A set S of vertices in graph G is a Neighbourhood set $(n - set)$ of G if $G_S$= $\bigcup_{u \in S}$ $<$

$(u) >$, where $< N(u) >$ is the subgraph induced by u and all vertices adjacent to $u \in S$, $\{u\}$ is not Neighbourhood set of G. The Neighbourhood number $n_\circ(G)$ of G is a minimum cardinality of a $n - set$ of G. This parameter is introduced by E. Sampathkumar and P. S. Neeralagi [6].

There are many types of domination numbers in literature [2]. Similarly we can define different types of Neighbourhood numbers by imposing certain conditions on Neighbourhood sets and derive some of the properties.

A Neighbourhood set S is said to be a maximal Neighbourhood set of G if the induced subgraph$< V - S >$ is not a Neighbourhood set of G. The maximal Neighbourhood number
$n_m(G)$ of G is the minimum cardinality of a maximal Neighbourhood set of G. This parameter is introduced by N.D. Soner et al [6].

In this chapter, we introduce the concept of SplitNeighbourhood as follows :

A Neighbourhood set S of a graph G is a Split Neighbourhood set if the induced subgraph$< V - S >$ is disconnected. The SplitNeighbourhood number $n_s(G)$ is the minimum cardinality of a SplitNeighbourhood set.

Thus, we observe that for any graph G,

$(G) \leq n_\circ(G) \leq n_s(G) \leq \alpha_\circ(G)$ ..................................... (I)

$(G) \leq \gamma_S(G) \leq n_s(G) \leq \alpha_\circ(G)$ .......................................(II)

Now we will prove the following results.

## 2. RESULTS

**Theorem A [4]** A dominating set D of G is a Split dominating set if and only if there exists two vertices $w_1, w_2 \in V - D$ such that $w_1 - w_2$ path contains a vertex of D.

**Theorem 2.1** For any graph G, $n_\circ(G) \leq n_s(G)$ ................................... (1)

Further the bound is attained if and only if there exists two vertices $w_1, w_2 \in V - S$such that every$w_1 - w_2$ path contains a vertex of S where S is a $n_\circ - set$ of G.

**Proof:** Equation (1) follows from the definition of SplitNeighbourhood set.

Further let S be a Neighbourhood set such that there exists two vertices $w_1, w_2 \in V - S$ such that every $w_1 - w_2$ path contains a vertex of S. Then $< V - S >$ is disconnected. Hence S is a SplitNeighbourhood set. This implies $n_s(G) \leq n_\circ(G)$. Then from (1) we have $n_\circ(G) = n_s(G)$.

Conversely suppose the bound is attained. Then if S is a Neighbourhood set, it isalso a SplitNeighbourhood set. This implies $< V - S >$ is disconnected. Hence there existtwo vertices $w_1, w_2 \in V - S$ such that every $w_1 - w_2$ path contains a vertex S.

**Theorem B [6] :**For a graph G, $n_\circ(G) = \gamma(G)$ if and only if there exists a minimum dominating set S. Such that every line in $< V - S >$ belongs to$< (u) >$ for some $u \in D$.

**Theorem 2.2** For any graph G,

$$\gamma_S(G) \leq n_s(G) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Further the bound is attained if and only if there exists a minimum Split dominating set S such that every line in $< V - S >$ belongs to $< N(u) >$ for some $u \in S$.

**Proof :**Since every SplitNeighbourhood set is a Split dominating set, hence Split dominating number is less than SplitNeighbourhood number. Suppose the bound is attained. This implies the condition is satisfied from Theorem 5.A [4].

Conversely, suppose that given condition is satisfied for some Split dominating set S. Then again by Theorem 5.B [6], S is a Neighbourhood set. Since $< V - S >$ is disconnected. S is a Split Neighbourhood set and hence from (2) the bound is attained.

**Theorem C [6]** For any graph G without isolated points,

$$\gamma(G) \leq n_\circ(G) \leq \alpha_\circ(G)$$

**Theorem 2.3** For any graph G without isolated points,

$$n_s(G) \leq \alpha_\circ(G) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Further the bound is attained if and only if there exist a Split Neighbourhood set S of G for which $V - S$ is independent with at least two vertices.

**Proof :**Let S be vertex cover of G. Then, $V - S$ is independent with at least two vertices. This implies, $< V - S >$ is disconnected. Also S is a Neighbourhood set from Theorem 5.C [6]. Hence S is a Split Neighbourhood set of G. This proves that the Split Neighbourhood number is less than or equal to vertex covering number.

Now to prove the second part, suppose there exist a Split Neighbourhood set S of Gfor which $V - S$ is independent with at least two vertices. This implies S is a vertex cover of G. Thus vertex covering number of G is less than or equal to the cardinality of S. Hence from (3), the bound is attained.

Conversely, suppose equality holds. Then there exists a Split Neighbourhood set S which is a vertex cover with $|S| = \alpha_\circ(G)$. Then obviously $V - S$ is independent with at least two vertices.

**Theorem D [4]** For any graph G, $\gamma \leq \gamma_s$

Hence from Theorem 5.1, 5.2, 5.3, 5.C [6] and 5.D [4]

we have,

$$(G) \leq n_\circ(G) \leq n_s(G) \leq \alpha_\circ(G) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{I})$$

$$(G) \leq \gamma_s(G) \leq n_s(G) \leq \alpha_\circ(G) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{II})$$

**Theorem 2.4** For any graph G,

$$(G) \leq n_s(G) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Where $(G)$ is the connectivity of graph G.

**Proof :**Let S be a Split Neighbourhood set of G. Then $< V - S >$ is disconnected.

Hence $(G) \leq n_s(G)$

Next, we list the exact value of $n_s(G)$ for some standard graphs

**Theorem 2.5** (i) For a path $P_n$ with n vertices,

$$n_s(P_n) = \left\lceil \frac{n}{3} \right\rceil \qquad n \geq 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

(ii) For a circle $C_n$with n vertices,

$$n_s(C_n) = \left\lceil \frac{n}{2} \right\rceil \qquad n \geq 4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

(iii) For a wheel $W_n$with n vertices,

$$n_s(W_n) = 3 \qquad n \geq 5 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

(iv) For a bipartite graph, without isolates, with bipartition $\{v_1, v_2\}$ of V(G),

$$n_s(G) \leq \min\{|v_1|, |v_2|\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Moreover the bound is attained by the graphs $K_{m,n}$

**Proof :**

(i)For a path $P_n$with n vertices where $n \geq 3$, every Neighbourhood set is a Split Neighbourhoodset. Hence (5) follows.

(ii) For a cycle $C_n$with n vertices where $n \geq 4$, every Neighbourhood set is a Split Neighbourhood set. Hence (6) follows.

(iii) For a wheel $W_n$with n vertices where $n \geq 5$, the vertex with degree $p - 1$ together with two non adjacent vertices on the cycle form a Split Neighbourhood set. Hence (7) follows.

(iv) For a bipartite graph with bipartition $\{V_1, V_2\}$of $V(G)$, both the sets with cardinality $V_1$ and $V_2$ are Split Neighbourhood sets. Hence (8) follows. Further if it is a complete bipartite graph then equality holds since for any $V_i$, i=1,2,3,…….

$V_i - \{u\}$is not a Split Neighbourhood set.

**Theorem E [6]** For any bipartite graph G without isolated points,

$$n_\circ(G) = \alpha_\circ(G) = \beta_1(G)$$

**Theorem 2.6** For any bipartite graph G without isolated points,

$$n_\circ(G) = n_s(G) = \alpha_\circ(G) = \beta_1(G) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (9)$$

**Proof :**This follows from Theorem 5.E [6] and Result (I)

**Theorem 2.7** A Split Neighbourhood set S is minimal if and only if for each vertex $v \in S$,one of the following conditions is satisfied

(i) $v$is an isolate in $< S >$

(ii) There exist a vertex $u \in V - S$ adjacent to $v$ but not adjacent to any vertex $w \in S$ adjacent to $v$.

(iii) $< (V - S) \cup \{v\} >$is connected.

**Proof:** Suppose $S$ is minimal, on the contrary, if there exists$v \in S$ such that $v$ does not satisfy any of the given conditions. Then $S' = S - \{v\}$ is a Neighbourhood set of G from (i) and (ii) and $< V - S' >$ is disconnected from (iii) This implies $S'$is Split Neighbourhood set of G. This is a contradiction. This proves that necessity.

Sufficiency is straight forward.

**Theorem F [1] :**For any non trivial connected graph G,

$$\alpha_\circ(G)+\beta_\circ(G) = p$$

**Theorem 2.8:**

i) For any graph G,

$$(G) \leq n_\circ(G) \leq n_s(G) \leq (\chi(G) - 1)\beta_\circ(G) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (10)$$

Provided$(G) \geq 2$, where $\chi(G)$ is the chromatic number of graph G.

ii) If G is bipartite graph which is not totally disconnected, Then,

$$(G) \leq n_{\circ}(G) \leq n_{s}(G) \leq \beta_{\circ}(G) \leq \chi(\bar{G}) \qquad (11)$$

Where $\bar{G}$ is complement of G.

**Proof :** Here we need to establish only the upper bound since lower bounds from I.

From Theorem 5.F [1] and the fact that $p \leq (G)(\beta_{\circ}(G))$

(See [1]) we have,

$$p - \beta_{\circ}(G) \leq \beta_{\circ}(G)(\chi(G) - 1)$$

$$\text{i.e.} \alpha_{\circ}(G) \leq \beta_{\circ}(G)(\chi(G) - 1)$$

Hence (10) follows from (1) and the fact that $\alpha_{\circ}(G) \leq \beta_{\circ}(G)(\chi(G) - 1)$

If G is bipartite, $(G) = 2$. Also (10) implies $n_{s}(G) \leq \beta_{\circ}(G)$

Hence (11) follows from the facts that $n_{s}(G) \leq \beta_{\circ}(G)$ and $\beta_{\circ}(G) \leq \chi(\bar{G})$ (See [1]).

**Theorem 2.9** For any graph G,

$$n_{s}(G) = 1 \qquad (12)$$

If and only if there exits a cut vertex with degree $p - 1$

**Proof :**Suppose$v$ is cutvertex of G of degree $p - 1$, then $\{v\}$ is a Neighbourhood set. Further since $< V - \{v\} >$ is disconnected. This implies $\{v\}$ is a Split Neighbourhood set. Hence $n_{s}(G) = 1$

Conversely, suppose $n_{s}(G) = 1$. Then, obviously there exists a cutvertex which is adjacent to all vertices. Hence there exists a cutvertex with degree $p - 1$.

**Theorem G [6]** For any $(p, q)$ graph G,

$$p - q + q_{\circ} \leq n_{\circ}(G) \leq p - \Delta(G)$$

$$]\frac{p}{\Delta(G)+1}] \leq n_{\circ}(G) \leq p - \beta_{\circ}(G) + p_{\circ}$$

Where $q_{\circ}$ =minimum {q(<D>;D is a minimal dominating set of G}

$p_{\circ}$ =the number of isolated vertices in G,

$\beta_{\circ}$ =set of independent vertices in G.

**Theorem 2.10** For any connected $(p, q)$ graph G,

$$p - q + q_{\circ} \leq n_s(G) \quad \text{.................................... (13)}$$

$$]\frac{p}{\Delta(G)+1}] \leq n_s(G) \leq p - \beta_{\circ}(G) \quad \text{..................... (14)}$$

**Proof :**The lower bounds in (13) and (14) follow from (1) and Theorem 5.G [6]. To prove upper bound in (14), we observe that $(V - M)$ is a Split Neighbourhood set where $M$is the set of $\beta_{\circ}$ independent points of G.

The lower bound in (13) and (14) is attained for the following graph in Figure 5

The upper bound in (14) is attained for any tree

The lower bound in (14) is attained by the following graph in figure 6.

**Theorem 2.11**

(i) $n_s(G) > p - \Delta(G)$if there exist a non-cutvertex of degree $p - 1$

(ii) $n_s(G) \leq p - \Delta(G)$if G has no triangle.

**Proof :**

(i) Let G has a non-cutvertex$v$ of degree $p - 1$. Then $\Delta(G) = p - 1$. Since $v$ is the non-cutvertex, $n_s(G) \geq 2$. Hence $n_s(G) > p - \Delta(G)$.

(ii) If G has no triangle then $n_s(G) \leq p - \Delta(G)$ from (9) and Theorem 5.G [6].

Now we obtain a Nordhaus-Gaddum type result.

**Theorem 2.12** Let G be a graph such that both $G$ and $\bar{G}$ are connected, then

$$n_s(G) + n_s(\bar{G}) \leq p(p-3) \quad \text{..............................} (15)$$

Further the bound is attained if and only if $G = P_4$

**Proof :**We have $n_s(G) \leq \alpha_\circ(G)$ from (3).

Since both $G$ and $\bar{G}$ are connected, $\Delta(G)$, $\Delta(\bar{G}) < p - 1$

This implies $\beta_\circ(G), \beta_\circ(\bar{G}) \geq 2$.

Hence $n_s(G) \leq p - 2$

$= 2(p-1) - p$

$\leq (2q - p)$

Similarly $n_s(\bar{G}) \leq 2\bar{q} - p$

Thus $n_s(G) + n_s(\bar{G}) \leq 2(q + \bar{q}) - 2p$

$\leq (p-1) - 2p$

$= (p-3)$

Suppose the bound is attained, then $n_s(G) = 2q - p$and $n_s(\bar{G}) = 2\bar{q} - p$. This implies $q$and $\bar{q} < p$. Hence and $\bar{G}$ are trees. i.e.$G = P_4$

Now we will establish a relation between Split Neighbourhood number and maximum Neighbourhood number.

**Theorem 2.13** LetG be a graph with $\beta_\circ(G) \geq 3$ and possess no triangles.

Then, $n_s(G) \leq n_m(G)$ ..................................................(16)

**Proof :**Let S be a maximal Neighbourhood set of $G$.Then $< V - S >$ is totally disconnected with at least two vertices. Thus S is a Split Neighbourhood set. Hence (16) holds.

**Theorem H [7]** For any graph $G$,

$$n_{\mathrm{m}}(G) \leq \alpha_{\circ}(G) + 1$$

**Theorem 2.14** Let$G$ be a graph without triangle, then

$$n_{\mathrm{m}}(G) \leq n_{\mathrm{s}}(G) + 1 \text{.......................................}(17)$$

**Proof :**The Proof of (17)follows from (9) and Theorem 5.H [7].

## REFERENCES


**[1]** F.Harary, **Graph-Theory,** Addison-Wesley Reading Mass, 1969.

**[2]** T.W.Haynes, S.T.Hedetniemi and P.J.Slater, **Fundamentals of Domination in Graphs**,Marcle Dekker, Inc, Newyork, 1997.

**[3]** V.R.Kulli and B.Janakiram, **The maximal domination number of a graph,** Graph Theory, Notes of New York, New York Academy of Sciences, **13:** 11-13, 1997.

**[4]** V.R.Kulli and Janakiram, **The split domination number of a Graph**, Graph Theory, Notes of New York Academy of Sciences

XXXII: 16-19, 1997.

**[5]** E.A. Nordhaus And J.W. Gaddum, **On complementary graphs**, Amer, Math Monthly, **63:** 175-77, 1956.

**[6]** E. Sampathkumar and P.S. Neeralagi**, The neighbourhood number of agraph**, Indian J. Pure Appl. Math, **16:** 126-132, 1985.

**[7]** N.D. Soner, B.ChaluvarajuAndB.Janakiram. **The maximal neighbourhood number of a graph**. Far East J. Appl. Math, **5:** 301-307, 2001.