The Ky Fan type inequalities for power exponential mean and its invariant

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Abstract: *In this paper, the investigations on mathematical means are carried out for the arguments lying on linear and curved paths. Further, an inequality chain involving important means is established which is an extension as well as strengthening of the Ky Fan type inequality, by using Taylor series expansion.*

Keywords: Ky Fan type inequality, linear and curved paths, means, Taylor series.

1. INTRODUCTION

The concept of Mathematical means is introduced and studied by Greek Mathematicians based on proportions and their importance in the fourth century A.D in the Pythagorean school [1, 4]. Later on several authors contributed and developed this field in view of the applications to various branches of science and technology. In recent years, Lokesha et al. obtained the relation between series and important means [9], Greek Means and functional means [3, 5-7, 14, 22-27], introduced and studied Gnan mean in two and n variables [3, 11], studied homogeneous functions as an application obtained some inequalities involving means [8, 10], firstly studied Oscillatory mean, Oscillatory type mean in Greek means, properties of new means, its generalizations and several mean inequality results were found in [2, 9, 11,12]. In [15-21], Nagaraja et al. established good number of inequalities involving means. In [28] Zhen Hang yang, proposed the Power exponential mean of the form: $Z(a,b)=(a^a b^b)^{\frac{1}{a+b}}$.

Definition 1.1: [13] For non-negative real numbers $y \in \left(0,\frac{1}{2}\right]$ and $(1-y) \in \left[\frac{1}{2},1\right)$ is represented as a function given below;

$$f(y)=\begin{cases} y, & \text{for } 0<y\le \frac{1}{2} \\ 1-y, & \text{for } \frac{1}{2}\le y<1 \end{cases}$$

In the discussion of the popular inequalities due to Ky Fan, the following are the standard notations in n variables.

For given n arbitrary non-negative real numbers $y_1, y_2, \dots y_n \in \left(0,\frac{1}{2}\right]$, the unweighted arithmetic, geometric and harmonic means are denoted by A_n, G_n and H_n respectively. Also, the arithmetic, geometric and harmonic means of the set of elements $1-y_1, 1-y_2, \dots, 1-y_n$ denoted by A_n', G_n' and H_n' respectively.

$A_n = A_n(y_1, y_2, \ldots, y_n) = \frac{1}{n}\sum_{i=1}^{n} y_i$

$G_n = G_n(y_1, y_2, \ldots, y_n) = (\prod_{i=1}^{n} y_i)^{\frac{1}{n}}$

$H_n = H_n(y_1, y_2, \ldots, y_n) = \frac{n}{\sum_{i=1}^{n}\left(\frac{1}{y_i}\right)}$

and

$A'_n = A'_n(1-y_1, 1-y_2, \ldots, 1-y_n) = \frac{1}{n}\sum_{i=1}^{n} 1-y_i$

$G'_n = G'_n(1-y_1, 1-y_2, \ldots, 1-y_n) = (\prod_{i=1}^{n} 1-y_i)^{\frac{1}{n}}$

$H'_n = H'_n(1-y_1, 1-y_2, \ldots, 1-y_n) = \frac{n}{\sum_{i=1}^{n}\left(\frac{1}{1-y_i}\right)}$

For two positive arguments e and f above said means are given by; $A = \frac{e+f}{2}$, $G = \sqrt{ef}$, $H = \frac{2ef}{e+f}$ and $A' = \frac{(1-e)+(1-f)}{2}$, $G' = \sqrt{(1-e)(1-f)}$, $H' = \frac{2(1-e)(1-f)}{(1-e)+(1-f)}$

The motivation of the work carried out by the eminent researchers and discussion with experts, results in the study of a function which is parabolic in nature, which is defined as follows.

Definition 1.2: For non-negative real numbers $y \in \left(0, \frac{1}{2}\right]$ and $(1-y) \in \left[\frac{1}{2}, 1\right)$ is represented as a parabolic function given below;

$$f^*(y) = \begin{cases} 2y^2, & \text{for } 0 < y \le \frac{1}{2} \\ 2(1-y)^2, & \text{for } \frac{1}{2} \le y < 1 \end{cases}$$

The functions $f(y)$ and $f^*(y)$ are graphically represented as shown below.

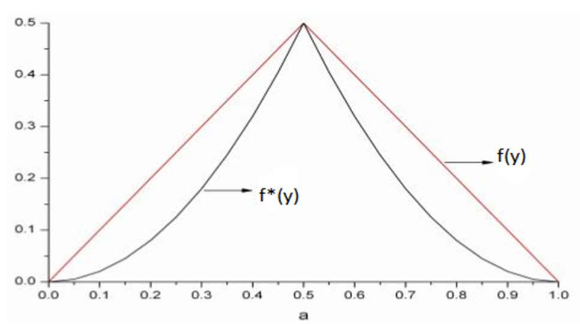

Figure 1. Graphical representation of $f(y)$ and $f^*(y)$

The means A^*, G^* and H^* are considered for the arguments lying on the curved path of $f^*(y)$ and some important inequality chains involving them are established.

2. EXPRESSIONS FOR MEANS WHOSE ARGUMENTS IN LINEAR PATH

Consider the function of the form; $f(y) = \begin{cases} y, & \text{for } 0 < y \le \frac{1}{2} \\ 1-y, & \text{for } \frac{1}{2} \le y < 1 \end{cases}$

For $a, b \in \left(0, \frac{1}{2}\right]$, then $Z(a,b) = \left(a^a b^b\right)^{\frac{1}{a+b}}$, $Z^i(a,b) = \left(a^b b^a\right)^{\frac{1}{a+b}}$, $A(a,b) = \frac{a+b}{2}$, $G(a,b) = \sqrt{ab}$, $H(a,b) = \frac{2ab}{a+b}$, $H_e(a,b) = \frac{a+\sqrt{ab}+b}{3}$, $C(a,b) = \frac{a^2+b^2}{a+b}$, and $C^i(a,b) = \frac{ab(a+b)}{a^2+b^2}$ are respectively called Power exponential mean, Invariant power exponential mean, Arithmetic mean, Geometric mean, Harmonic mean, Heron mean, Contra harmonic mean and invariant contra harmonic mean.

For $a' = 1-a,\ b' = 1-b \in \left[\frac{1}{2}, 1\right)$ the above said means are given by;

$Z'(a,b) = \left[(1-a)^{(1-a)}(1-b)^{(1-b)}\right]^{\frac{1}{2-(a+b)}}$;

$\left(Z^i\right)'(a,b) = \left[(1-a)^{(1-b)}(1-b)^{(1-a)}\right]^{\frac{1}{2-(a+b)}}$

$A'(a,b) = \frac{2-(a+b)}{2}$; $G'(a,b) = \sqrt{1-a-b+ab}$; $H'(a,b) = \frac{2(1-a-b+ab)}{2-(a+b)}$

$H_e'(a,b) = \frac{2-a-b+\sqrt{1-a-b+ab}}{3}$; $C'(a,b) = \frac{2+a^2+b^2-2(a+b)}{2-(a+b)}$; and

$$\left(C^i\right)'(a,b) = \frac{1-a-b+ab\big(2-(a+b)\big)}{2+a^2+b^2-2(a+b)}$$

3. NEW FUNCTIONS AND THEIR TAYLOR' SERIES EXPANSION

Now, define the ratio of means in the form of functions such that numerator is a mean for arguments in $\left(0, \frac{1}{2}\right]$ and denominator is a mean for arguments in $\left[\frac{1}{2}, 1\right)$as follows;

$$f_1(a,b) = \frac{z^{i(a,b)}}{\left(z^i\right)^1(a,b)} = \frac{\left(a^b b^a\right)^{\frac{1}{a+b}}}{\left[(1-a)^{(1-b)}(1-b)^{(1-a)}\right]^{\frac{1}{2-(a+b)}}}$$

$$f_2(a,b) = \frac{H(a,b)}{H^1(a,b)} = \frac{\dfrac{ab}{a+b}}{\dfrac{1-b-a+ab}{2-(a+b)}}$$

$$f_3(a,b) = \frac{G(a,b)}{G^1(a,b)} = \frac{\sqrt{ab}}{\sqrt{1-a-b+ab}}$$

$$f_4(a,b) = \frac{H_e(a,b)}{H_e^1(a,b)} = \frac{a+\sqrt{ab}+b}{2-a-b+\sqrt{1-a-b+ab}}$$

$$f_5(a,b) = \frac{A(a,b)}{A^1(a,b)} = \frac{a+b}{2-(a+b)}$$

$$f_6(a,b) = \frac{Z(a,b)}{Z^1(a,b)} = \frac{\left(a^a . b^b\right)^{\frac{1}{a+b}}}{\left[(1-a)^{(1-a)}(1-b)^{(1-b)}\right]^{\frac{1}{2-(a+b)}}}$$

$$f_7(a,b) = \frac{C(a,b)}{C^1(a,b)} = \frac{\dfrac{a^2+b^2}{a+b}}{\dfrac{2+a^2+b^2-2(a+b)}{2-(a+b)}}$$

$$f_8(a,b) = \frac{C^i(a,b)}{\left(C^i\right)^1(a,b)} = \frac{\dfrac{ab(a+b)}{a^2+b^2}}{\dfrac{1-a-b+ab\big(2-(a+b)\big)}{2+a^2+b^2-2(a+b)}}$$

Put $b=\frac{1}{2}$ and $a=t\in\left(0,\frac{1}{2}\right]$, then Taylor's series expansion of above defined functions is given by;

$$f_1\left(t,\frac{1}{2}\right)=1+2\left(t-\frac{1}{2}\right)+2\left(t-\frac{1}{2}\right)^2+8\left(t-\frac{1}{2}\right)^3+14\left(t-\frac{1}{2}\right)^4+\cdots$$

$$f_2\left(t,\frac{1}{2}\right)=1+2\left(t-\frac{1}{2}\right)+2\left(t-\frac{1}{2}\right)^2+6\left(t-\frac{1}{2}\right)^3+10\left(t-\frac{1}{2}\right)^4+\cdots$$

$$f_3\left(t,\frac{1}{2}\right)=1+2\left(t-\frac{1}{2}\right)+2\left(t-\frac{1}{2}\right)^2+4\left(t-\frac{1}{2}\right)^3+6\left(t-\frac{1}{2}\right)^4+\cdots$$

$$f_4\left(t,\frac{1}{2}\right)=1+2\left(t-\frac{1}{2}\right)+2\left(t-\frac{1}{2}\right)^2+\frac{8}{3}\left(t-\frac{1}{2}\right)^3+\frac{10}{3}\left(t-\frac{1}{2}\right)^4+\cdots$$

$$f_5\left(t,\frac{1}{2}\right)=1+2\left(t-\frac{1}{2}\right)+2\left(t-\frac{1}{2}\right)^2+2\left(t-\frac{1}{2}\right)^3+2\left(t-\frac{1}{2}\right)^4+\cdots$$

$$f_6\left(t,\frac{1}{2}\right)=1+2\left(t-\frac{1}{2}\right)+2\left(t-\frac{1}{2}\right)^2-2\left(t-\frac{1}{2}\right)^3-\frac{20}{3}\left(t-\frac{1}{2}\right)^4+\cdots$$

$$f_7\left(t,\frac{1}{2}\right)=1+2\left(t-\frac{1}{2}\right)+2\left(t-\frac{1}{2}\right)^2-2\left(t-\frac{1}{2}\right)^3-6\left(t-\frac{1}{2}\right)^4+\cdots$$

$$f_8\left(t,\frac{1}{2}\right)=1+2\left(t-\frac{1}{2}\right)+2\left(t-\frac{1}{2}\right)^2+10\left(t-\frac{1}{2}\right)^3+18\left(t-\frac{1}{2}\right)^4+\cdots$$

It was observed that up to 2nd degree, the terms of all the eight series are same. By considering 3rd and 4th degree terms, the following graph interpret the eight means.

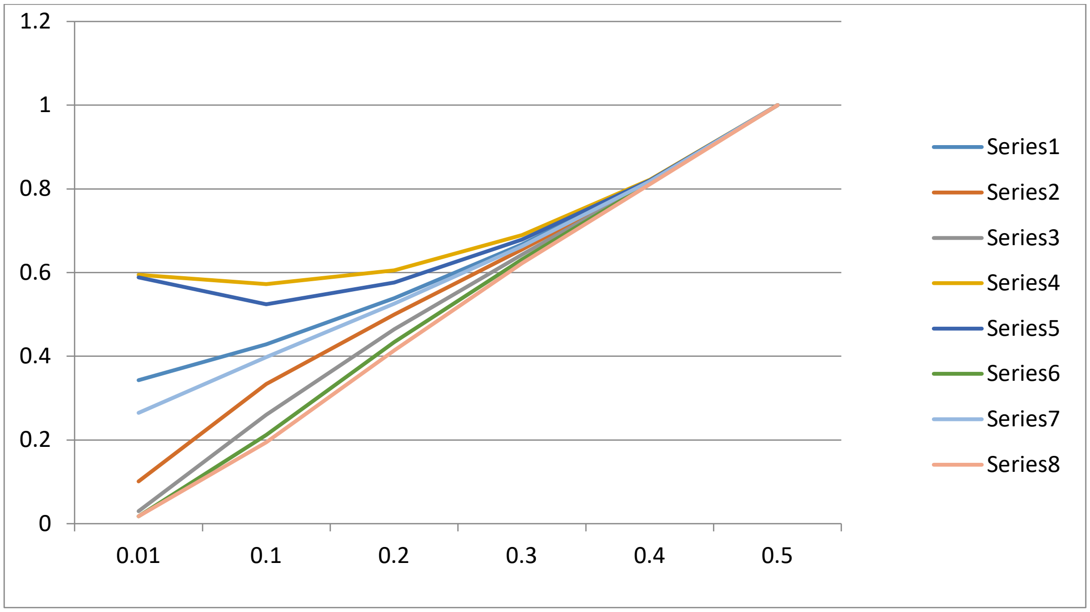

Figure 2. Graphical representation of f_1 to f_8

It was observed that $\frac{Z(a,b)}{Z^1(a,b)} < \frac{C(a,b)}{C^1(a,b)}$ is holds for $t \in (0.0076096, 0.5)$ and $\frac{C(a,b)}{C^1(a,b)} < \frac{Z(a,b)}{Z^1(a,b)}$ is holds for $t \in (0.00001, 0.0076096)$, also, it is evident from the graphs see (figure 2 and figure 3).

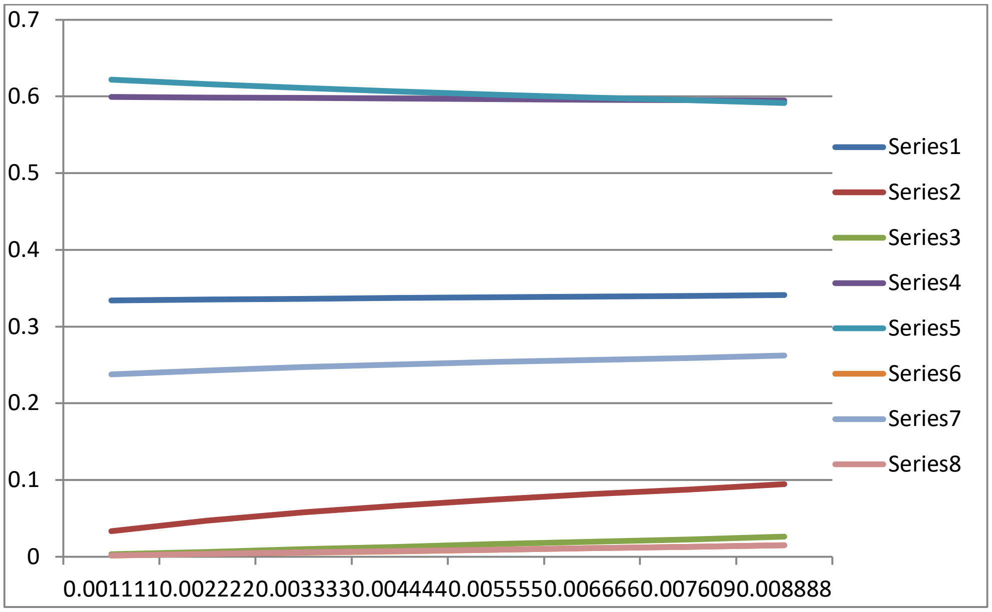

Figure 3. Graphical representation of f_1 to f_8

Figure 3 shows the graph of f_1 to f_8 which holds for $t \in (0.00001, 0.0076096)$ and Figure 2 shows the graph of f_1 to f_8 which holds for $t \in (0.0076096, 0.5)$, from the above facts the following interpolating Ky-Fan type inequality chain holds. Thus, the following two theorems are concluded.

Theorem 1: For $b = \frac{1}{2}$ and $a = t \in (0.0076096, 0.5)$, then

$$\frac{C^i(a,b)}{(C^i)^1(a,b)} < \frac{Z^i(a,b)}{(Z^i)^1(a,b)} < \frac{H(a,b)}{H^1(a,b)} < \frac{G(a,b)}{G^1(a,b)} < \frac{H_e(a,b)}{H_e^1(a,b)} < \frac{A(a,b)}{A^1(a,b)} < \frac{Z(a,b)}{Z^1(a,b)} < \frac{C(a,b)}{C^1(a,b)}$$

Theorem 2: For $b = \frac{1}{2}$ and $a = t \in (0.00001, 0.0076096)$, then

$$\frac{C^i(a,b)}{(C^i)^1(a,b)} < \frac{Z^i(a,b)}{(Z^i)^1(a,b)} < \frac{H(a,b)}{H^1(a,b)} < \frac{G(a,b)}{G^1(a,b)} < \frac{H_e(a,b)}{H_e^1(a,b)} < \frac{A(a,b)}{A^1(a,b)} < \frac{C(a,b)}{C^1(a,b)} < \frac{Z(a,b)}{Z^1(a,b)}$$

4. EXPRESSIONS FOR MEANS WHOSE ARGUMENTS IN CURVED PATH

Consider the function of the form; $f^*(y) = \begin{cases} 2y^2, & \text{for } 0 < y \leq \frac{1}{2} \\ 2(1-y)^2, & \text{for } \frac{1}{2} \leq y < 1 \end{cases}$

For $a, b \in \left(0, \frac{1}{2}\right]$, then $Z^*(a,b) = \left[(2a^2)^{2a^2}(2b^2)^{2b^2}\right]^{\frac{1}{2a^2+2b^2}}$

$Z^{i*}(a,b) = \left[(2a^2)^{2b^2}(2b^2)^{2a^2}\right]^{\frac{1}{2a^2+2b^2}}$; $A^*(a,b) = \frac{2a^2+2b^2}{2}$; $G^*(a,b) = \sqrt{2a^2 2b^2}$

$H^*(a,b) = \frac{4a^2b^2}{a^2+b^2}$; $H_e^*(a,b) = \frac{2a^2+\sqrt{4a^2b^2}+2b^2}{3}$; $C^*(a,b) = \frac{2a^4+2b^4}{a^2+b^2}$ and $C^{i*} = \frac{2a^2b^2(a^2+b^2)}{a^4+b^4}$ are respectively called power exponential mean, invariant power exponential mean, Arithmetic mean, Geometric mean, Harmonic mean, Heron mean, Contra harmonic mean and invariant contra harmonic mean.

For $a' = 1-a,\ b' = 1-b \in \left[\frac{1}{2},1\right)$ the above said means are given by;

$$(Z^*)'(a,b) = \left[(2(1-a)^2)^{2(1-a)^2}(2(1-b)^2)^{2(1-b)^2}\right]^{\frac{1}{2(1-a)^2+2(1-b)^2}};$$

$$(Z^{i*})'(a,b) = \left[(2(1-a)^2)^{2(1-b)^2}(2(1-a)^2)^{2(1-b)^2}\right]^{\frac{1}{2(1-a)^2+2(1-b)^2}};$$

$$(A^*)'(a,b) = \frac{2(1-a)^2+2(1-b)^2}{2}; \qquad (G^*)'(a,b) = \sqrt{2(1-a)^2 2(1-b)^2}$$

$$(H^*)'(a,b) = \frac{4(1-a)^2(1-b)^2}{(1-a)^2+(1-b)^2}; \qquad (H_e^*)'(a,b) = \frac{2(1-a)^2+\sqrt{4(1-a)^2(1-b)^2}+2(1-b)^2}{3};$$

$$(C^*)'(a,b) = \frac{2(1-a)^4+2(1-b)^4}{(1-a)^2+(1-b)^2}; \text{ and } \qquad (C^{i*})'(a,b) = \frac{2(1-a)^2(1-b)^2\left[(1-a)^2(1-b)^2\right]}{(1-a)^4+(1-b)^4}$$

5. NEW FUNCTIONS AND THEIR TAYLOR' SERIES EXPANSION

Now, define the ratio of means in the form of functions such that numerator is a mean for arguments in $\left(0,\frac{1}{2}\right]$ and denominator is a mean for arguments in $\left[\frac{1}{2},1\right)$ as follows;

$$f_1^*(a,b) = \frac{Z^{i*}(a,b)}{(Z^{i*})^1(a,b)} = \frac{\left[(2a^2)^{(2b^2)}(2b^2)^{2a^2}\right]^{\frac{1}{2a^2+2b^2}}}{\left[(2(1-a)^2)^{2(1-b)^2}(2(1-a)^2)^{2(1-b)^2}\right]^{\frac{1}{2(1-a)^2+2(1-b)^2}}}$$

$$f_2^*(a,b) = \frac{H^*(a,b)}{(H^*)^1(a,b)} = \frac{\left\{\frac{4a^2b^2}{a^2+b^2}\right\}}{\left(\frac{4(1-a)^2(1-b)^2}{(1-a)^2+(1-b)^2}\right)}$$

$$f_3^*(a,b) = \frac{G^*(a,b)}{(G^*)^1(a,b)} = \frac{\sqrt{2a^2 2b^2}}{\sqrt{2(1-a)^2 2(1-b)^2}}$$

$$f_4^*(a,b) = \frac{H_e^*(a,b)}{(H_e^*)^1(a,b)} = \frac{\frac{2a^2+\sqrt{4a^2b^2}+2b^2}{3}}{\frac{2(1-a)^2+\sqrt{4(1-a)^2(1-b)^2}+2(1-b)^2}{3}}$$

$$f_5^*(a,b) = \frac{A^*(a,b)}{(A^*)^1(a,b)} = \frac{\frac{2a^2+2b^2}{2}}{\frac{2(1-a)^2+2(1-b)^2}{2}}$$

$$f_6^*(a,b) = \frac{Z^*(a,b)}{(Z^*)^1(a,b)} = \frac{\left[(2a^2)^{2a^2}(2b^2)^{2b^2}\right]^{\frac{1}{2a^2+2b^2}}}{\left[(2(1-a)^2)^{2(1-a)^2}(2(1-b)^2)^{2(1-b)^2}\right]^{\frac{1}{2(1-a)^2+2(1-b)^2}}}$$

$$f_7^*(a,b) = \frac{C^*(a,b)}{(C^*)^1(a,b)} = \frac{\frac{2a^4+2b^4}{a^2+b^2}}{\frac{2(1-a)^4+2(1-b)^4}{(1-a)^2+(1-b)^2}}$$

$$f_8^*(a,b) = \frac{C^{i*}(a,b)}{(C^{i*})^1(a,b)} = \frac{\frac{2a^2b^2(a^2+b^2)}{a^4+b^4}}{\frac{2(1-a)^2(1-b)^2[(1-a)^2(1-b)^2]}{(1-a)^4+(1-b)^4}}$$

Put $b = \frac{1}{2}$ and $a = t \in \left(0, \frac{1}{2}\right]$, then Taylor's series expansion of above defined functions is given by;

$$f_1^*\left(t,\frac{1}{2}\right) = 1 + 4\left(t-\frac{1}{2}\right) + 8\left(t-\frac{1}{2}\right)^2 + 32\left(t-\frac{1}{2}\right)^3 + 96\left(t-\frac{1}{2}\right)^4 + \cdots$$

$$f_2^*\left(t,\frac{1}{2}\right) = 1 + 4\left(t-\frac{1}{2}\right) + 8\left(t-\frac{1}{2}\right)^2 + 24\left(t-\frac{1}{2}\right)^3 + 64\left(t-\frac{1}{2}\right)^4 + \cdots$$

$$f_3^*\left(t,\frac{1}{2}\right) = 1 + 4\left(t-\frac{1}{2}\right) + 8\left(t-\frac{1}{2}\right)^2 + 16\left(t-\frac{1}{2}\right)^3 + 32\left(t-\frac{1}{2}\right)^4 + \cdots$$

$$f_4^*\left(t,\frac{1}{2}\right) = 1 + 4\left(t-\frac{1}{2}\right) + 8\left(t-\frac{1}{2}\right)^2 + \frac{32}{3}\left(t-\frac{1}{2}\right)^3 + \frac{32}{3}\left(t-\frac{1}{2}\right)^4 + \cdots$$

$$f_5^*\left(t,\frac{1}{2}\right) = 1 + 4\left(t-\frac{1}{2}\right) + 8\left(t-\frac{1}{2}\right)^2 + 8\left(t-\frac{1}{2}\right)^3 - 16\left(t-\frac{1}{2}\right)^4 + \cdots$$

$$f_6^*\left(t,\frac{1}{2}\right) = 1 + 4\left(t-\frac{1}{2}\right) + 8\left(t-\frac{1}{2}\right)^2 + 0\left(t-\frac{1}{2}\right)^3 - 32\left(t-\frac{1}{2}\right)^4 + \cdots$$

$$f_7^*\left(t,\frac{1}{2}\right) = 1 + 4\left(t-\frac{1}{2}\right) + 8\left(t-\frac{1}{2}\right)^2 - 8\left(t-\frac{1}{2}\right)^3 - 64\left(t-\frac{1}{2}\right)^4 + \cdots$$

$$f_8^*\left(t,\frac{1}{2}\right) = 1 + 4\left(t-\frac{1}{2}\right) + 8\left(t-\frac{1}{2}\right)^2 + 40\left(t-\frac{1}{2}\right)^3 + 128\left(t-\frac{1}{2}\right)^4 + \cdots$$

It was observed that up to 2nd degree term all the terms of the series are same. By considering 3rd and 4th degree terms, the following interpolating Ky-fan type inequality chain holds.

Theorem 3: For $b = \frac{1}{2}$ and $a = t \in \left(0, \frac{1}{2}\right]$, then

$$\frac{C^{i*}(a,b)}{(C^{i*})^1(a,b)} < \frac{Z^{i*}(a,b)}{(Z^{i*})^1(a,b)} < \frac{H^*(a,b)}{(H^*)^1(a,b)} < \frac{G^*(a,b)}{(G^*)^1(a,b)}$$
$$< \frac{H_e^*(a,b)}{(H_e^*)^1(a,b)} < \frac{A^*(a,b)}{(A^*)^1(a,b)} < \frac{Z^*(a,b)}{(Z^*)^1(a,b)} < \frac{C^*(a,b)}{(C^*)^1(a,b)}$$

The graphical representation of theorem 3 is as follows:

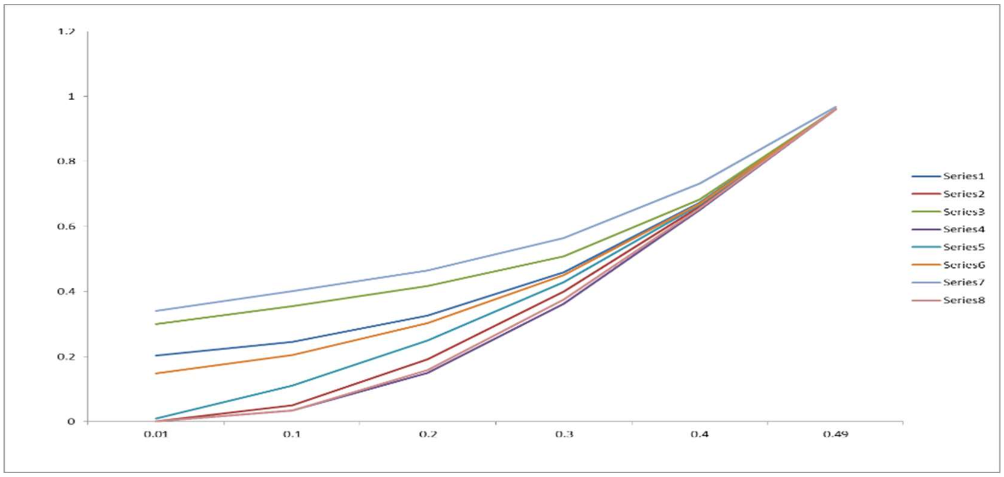

Figure 4. Graphical representation of f_1^* to f_8^*

The theorems 1 and 3 are combined together and the resulting interpolating inequality chain is gives as below.

Theorem 4: For $b = \frac{1}{2}$ and $a = t \in (0.0076096, 0.5)$, then

$$\frac{C^i(a,b)}{(C^i)^1(a,b)} < \frac{Z^i(a,b)}{(Z^i)^1(a,b)} < \frac{H(a,b)}{H^1(a,b)} < \frac{G(a,b)}{G^1(a,b)} < \frac{H_e(a,b)}{H_e^1(a,b)} < \frac{A(a,b)}{A^1(a,b)} < \frac{Z(a,b)}{Z^1(a,b)} < \frac{C(a,b)}{C^1(a,b)} < \frac{C^{i*}(a,b)}{(C^{i*})^1(a,b)} < \frac{Z^{(i*)(a,b)}}{(Z^{i*})^1(a,b)} < \frac{H^*(a,b)}{(H^*)^1(a,b)} < \frac{G^*(a,b)}{(G^*)^1(a,b)} < \frac{H_e^*(a,b)}{(H_e^*)^1(a,b)} < \frac{A^*(a,b)}{(A^*)^1(a,b)} < \frac{Z^*(a,b)}{(Z^*)^1(a,b)} < \frac{C^*(a,b)}{(C^*)^1(a,b)}$$

The theorems 2 and 3 are combined together and the resulting interpolating inequality chain is gives as below.

Theorem 5: For $b = \frac{1}{2}$ and $a = t \in (0.00001, 0.0076096)$, then

$$\frac{C^i(a,b)}{(C^i)^1(a,b)} < \frac{Z^i(a,b)}{(Z^i)^1(a,b)} < \frac{H(a,b)}{H^1(a,b)} < \frac{G(a,b)}{G^1(a,b)} < \frac{H_e(a,b)}{H_e^1(a,b)} < \frac{A(a,b)}{A^1(a,b)} < \frac{C(a,b)}{C^1(a,b)} < \frac{Z(a,b)}{Z^1(a,b)} < \frac{C^{i*}(a,b)}{(C^{i*})^1(a,b)} < \frac{Z^{(i*)(a,b)}}{(Z^{i*})^1(a,b)} < \frac{H^*(a,b)}{(H^*)^1(a,b)} < \frac{G^*(a,b)}{(G^*)^1(a,b)} < \frac{H_e^*(a,b)}{(H_e^*)^1(a,b)} < \frac{A^*(a,b)}{(A^*)^1(a,b)} < \frac{Z^*(a,b)}{(Z^*)^1(a,b)} < \frac{C^*(a,b)}{(C^*)^1(a,b)}$$

Note: The inequality chain so established for two arguments is holds good for n arguments.

Conclusion: In this paper, new inequalities on ratio of means are established which are similar to Ky Fan type inequalities by considering triangular wave function and parabolic wave function in the interval (0,1). Further, investigation to be carried out on more number of parabolic wave functions based on their nature.

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