SUB-COMPATIBLE MAPPINGS AND FIXED POINT THEOREMS IN INTUITIONISTIC FUZZY METRIC SPACE

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ABSTRACT

In this paper, common fixed point theorems for sub-compatible and sub-sequentially continuous mappings in intuitionistic fuzzy metric space has been proved which is a generalization of the result of Turkoglu et. al. [11]. We also cited an example in support of our result.

Keywords : Common fixed points, intuitionistic fuzzy metric space, compatible maps and sub-compatible mappings.

AMS Subject Classification : Primary 47H10, Secondary 54H25.

1. INTRODUCTION

The concept of fuzzy set has been investigated by Zadeh's [12] which led to a rich growth of fuzzy Mathematics. Many authors have introduced the concept of fuzzy metric in different ways. Atanassov [3] introduced and studied the concept of intuitionistic fuzzy set. In 2004, the notion of intuitionstic fuzzy metric space defined by Park [10] is a generalization of fuzzy metric space due to George and Veeramani [5]. Afterwards, using the idea of Intuitionistic Fuzzy set, Alaca et.al. [1] defined the notion of Intuitionistic Fuzzy Metric space, as Park [10] with the help of continuous t – norms and continuous t–conorms, as a generalization of fuzzy metric space due to Kramosil and Michalek [7], further Coker [4], Turkoglu [11] and others have been expansively developed the theory of Intuitionistic Fuzzy set and applications. After generalizing the Jungck's [6] common fixed point theorem in intuitionistic fuzzy metric, Turkoglu et. al. [11] introduced the notion of Cauchy sequences in intuitionistic fuzzy metric space space and proved the intuitionistic fuzzy version of Pant's theorem [9] by giving the definition of weakly commuting and R-weakly commuting mappings in intuitionistic fuzzy metric space. For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.

2. PRELIMINARIES

Definition 2.1. [1] A binary operation $*:[0,1] \times [0, 1] \rightarrow [0,1]$ is continuous t-norm if * satisfies the following conditions :

(i) * is commutative and associative;

(ii) * is continuous;

(iii) $a * 1 = a$ for all $a \in [0,1]$;

(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 2.2. [1] A binary operation $\Diamond: [0,1] \times [0,1] \rightarrow [0,1]$ is continuous t-conorm if satisfies the following conditions :

(i) $\Diamond$ is commutative and associative;

(ii) $\Diamond$ is continuous;

(iii) $a \Diamond 0 = a$ for all $a \in [0,1]$;

(iv) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$

for all $a, b, c, d \in [0,1]$.

Remark 2.1. [1] The concepts of triangular norms (t-norms) and triangular co-norms (t-conorms) are known as axiomatic skeletons that we use for characterizing fuzzy intersections and unions respectively.

Definition 2.3. [1] A 5-tuple (X,M,N,*,) is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, * is a continuous t-norm, is a continuous t-conorm and M, N are fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions :

(i) $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$;

(ii) $M(x, y, 0) = 0$ for all $x, y \in X$;

(iii) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;

(iv) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(v) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$ for all $x, y, z \in X$ and $s, t > 0$;

(vi) for all $x, y \in X$, $M(x, y, .) : [0, \infty) \rightarrow [0,1]$ is left continuous;

(vii) $\lim_{t\to\infty} M(x, y, t) = 1$ for all x, y ∈X and t > 0;

(viii) N(x, y, 0) = 1 for all x, y ∈X;

(ix) N(x, y, t) = 0 for all x, y ∈ X and t > 0 if and only if x = y;

(x) N(x, y, t) = N(y, x, t) for all x, y ∈ X and t > 0;

(xi) N(x, y, t) ◊ N(y, z, s) ≥ N(x, z, t + s) for all x, y, z ∈X and s, t > 0;

(xii) for all x, y ∈X, N(x, y, .) : [0,∞) → [0,1] is right continuous ;

(xiii) $\lim_{t\to\infty} N(x, y, t) = 0$ for all x, y in X.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions M(x,y,t) and N(x,y,t) denote the degree of nearness and the degree of non nearness between x and y with respect to t, respectively.

Remark 2.2. [1] Every fuzzy metric space (X, M, *) is an intuitionistic fuzzy metric space of the form (X, M, 1-M, *, ◊) such that t-norm * and t-conorm are associated, i.e,

$$x \lozenge y = 1- (1 - x) * (1 - y) \quad \text{for all } x, y \in X.$$

Example 2.1. [1] Let (X , d) be a metric space. Define t -norm a * b = min{a,b} and t -conorm a ◊ b = max{a, b}) and for all x, y ∈ X and t > 0 ,

$$M_d(x,y,t) = \frac{t}{t + d(x,y)}, \quad N_d(x,y,t) = \frac{d(x,y)}{t + d(x,y)}.$$

Then (X, M,N, *, ◊) is an IFM-space and the intuitionistic fuzzy metric (M, N) induced by the metric d is often referred to as the standard intuitionistic fuzzy metric.

Remark 2.3. [1] In intuitionistic fuzzy metric space X, M(x, y, .) is non-decreasing and N(x, y, .) is non-increasing for all x, y ∈ X.

Definition 2.4. [1] Let (X, M, N, *, ◊) be an intuitionistic fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all t > 0 and p > 0,

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0.$$

(b) a sequence $\{x_n\}$ in X is said to be convergent to a point x ∈ X if, for all t > 0,

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(x_n, x, t) = 0$$

Definition 2.5. [1] An intuitionistic fuzzy metric space (X, M, N, *, ◊) is said to be complete if and only if every Cauchy sequence in X is convergent.

Lemma 2.1. [3] Let (X, M, N, *, ◊) be an intuitionistic fuzzy metric space. If there exists $k \in (0,1)$ such that

$$M(x,y,kt) \geq M(x,y,t) \text{ and } N(x,y,kt) \leq N(x,y,t) \text{ for } x, y \in X.$$

Then x = y.

Lemma 2.2. [3] Let (X, M, N, *, ◊) be an intuitionistic fuzzy metric space. If there exists a number $k \in (0,1)$ such that

(a) $M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$, $N(y_{n+2}, y_{n+1}, kt) \leq N(y_{n+1}, y_n, t)$

for all t > 0 and n = 1, 2, ... then $\{y_n\}$ is a Cauchy sequence in X.

Definition 2.6.[1] Two maps A and B from an intuitionistic fuzzy metric space (X, M, N, *, ◊) into itself are said to compatible if

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0$$

for all t > 0, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \qquad \text{for some } x \in X.$$

Definition 2.7. [2] Two maps A and B from an intuitionistic fuzzy metric space (X, M, N, *, ◊) into itself are said to occasionally weakly compatible if and only if there is a point x in X which is coincidence point of A and B at which A and B commute.

Definition 2.8. [8] Two self maps A and B of an intuitionistic fuzzy metric space (X, M, N, *, ◊) are said to be sub-compatible if and only if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,\ z \in X$ and which satisfy the condition

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0.$$

Remark 2.4. Two occasionally weakly compatible maps are sub-compatible maps, but the converse is not true as shown in the following example.

Example 2.2. Let X = R be the set of real numbers with * continuous t-norm and ◊ continuous t-conorm defined by a * b = ab and a ◊ b = min {1, a+ b} respectively, for all a, b ∈ [0,1].

Define A,B: $X \to X$ as;

$$Ax = \begin{cases} x, x < 1 \\ 2x - 1, x \geq 1 \end{cases}, Bx = \begin{cases} 4x - 3, x < 1 \\ x + 3, x \geq 1 \end{cases}$$

Define a sequence $x_n = 1 - \frac{1}{n}$, then $Ax_n = \left(1 - \frac{1}{n}\right) \to 1$

$$Bx_n = 4\left(1 - \frac{1}{n}\right) - 3 = 1 - \frac{4}{n} \to 1$$

$$ABx_n = A\left(1 - \frac{4}{n}\right) = 1 - \frac{4}{n}$$

$$BAx_n = B\left(1 - \frac{1}{n}\right) = 4\left(1 - \frac{1}{n}\right) - 3 = 1 - \frac{4}{n} \text{ and}$$

$\lim_{n\to\infty} M(ABx_n, BAx_n, t)) \to 0$ and $\lim_{n\to\infty} N(ABx_n, BAx_n, t)) \to 0$.

Thus, A and B are sub compatible but A and B are not occasionally weakly compatible maps as, A (4) = 7 = B (4)

and AB (4) = A (7) =13 $\neq$ BA (4) =10 .

Definition 2.9. [8] Two self maps A and B of an intuitionistic fuzzy metric space (X, M, N, *, ◊) are said to be sub sequentially continuous if and only if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some t $\in X$ and satisfy $\lim_{n\to\infty} ABx_n = At$ and $\lim_{n\to\infty} BAx_n = Bt$.

In 2006, Turkoglu et. al. [11] proved the following :

Theorem 2.1. [11] Let (X, M, N, *, ◊) be a complete intuitionistic fuzzy metric space. Let f, g : X $\to$ X be self mappings satisfying the following conditions :

(i) $g(X) \subset f(X)$;

(ii) f is continuous;

(iii) there exists a number k $\in$ (0, 1) such that

$$M(g(x), g(y), kt) \geq M(f(x), f(y), t)$$

$$N(g(x), g(y), kt) \leq N(f(x), f(y), t)$$

for all x, y $\in$X and t > 0.

Then f and g have a unique common fixed point in X provided f and g commute.

3. MAIN RESULT.

In this section, we prove a common fixed point theorem for sub-compatible and sub-sequentially continuous mappings in intuitionistic fuzzy metric space which is a generalization of Turkoglu et. al. [11].

Theorem 3.1. Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric spaces. Let $f, g : X \to X$ be sub-compatible and sub-sequential mappings satisfying the following conditions :

(3.1) $g(X) \subset f(X)$;

(3.2) there exists a number $k \in (0, 1)$ such that

$$M(gx, gy, kt) \geq M(fx, fy, t)$$

$$N(gx, gy, kt) \leq N(fx, fy, t)$$

for all $x, y \in X$ and $t > 0$;

(3.3) one of the subspace $g(X)$ or $f(X)$ is complete.

Then f and g have a unique common fixed point in X.

Proof. By (3.1), since $g(X) \subset f(X)$, for any $x_0 \in X$, there exists a point $x_1 \in X$ such that $gx_0 = fx_1$.

In general, chose x_{n+1} such that

$$y_n = fx_{n+1} = gx_n.$$

From Turkoglu et. al. [11], we conclude that $\{y_n\}$ is a Cauchy sequence in X.

Since either f(X) or g(X) is complete, for definiteness assume that f(X) is complete.

Since f(X) is complete, so there exists a point $p \in X$ such that

$$fp = z.$$

Now using (3.2), we have

$$M(gp, gx_n, kt) \geq M(fp, fx_n, t)$$

and $$N(gp, gx_n, kt) \leq N(fp, fx_n, t).$$

Taking limit as $n \to \infty$, we obtain

$$gp = z.$$

Therefore, we have fp = gp =z.
If the pair (f, g) is sub-compatible and subsequentially continuous then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} gx_{2n} = z$ for some $z \in X$ and which satisfy

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = M(fz, gz, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} N(fgx_n, gfx_n, t) = N(fz, gz, t) = 0$$

i.e. fz = gz .

Now we show that z is a common fixed point of f and g.

From (3.2), we get

$$M(gz, gx_n, kt) \geq M(fz, fx_n, t)$$

and $$N(gz, gx_n, kt) \leq N(fz, fx_n, t)$$

Proceeding limit as $n\to\infty$, we obtain

$$gz = z.$$

Hence z is a common fixed point of f and g both.

Uniqueness : Let w be another common fixed point of f and g then

$$fw = gw = w.$$

Using (3.2), we get

$$M(gz, gw, kt) \geq M(fz, fw, t)$$

$$N(gz, gw, kt) \leq N(fz, fw, t)$$

or $$M(z, w, kt) \geq M(z, w, t)$$

$$N(z, w, kt) \leq N(z, w, t).$$

Using lemma 2.1, we get

$$z = w.$$

Hence, z is the unique common fixed point of f and g.

This completes the proof.

Example 3.1. Let $X = \left\{\frac{1}{n} : n \in N\right\} \cup \{0\}$ with * continuous t-norm and ◊ continuous t-conorm defined by a * b = ab and a ◊ b = min {1, a+ b} respectively, for all a, b ∈ [0,1]. For each t ∈ [0, ∞) and x, y ∈ X, define (M, N) by

$$M(x,y,t)=\begin{cases}\dfrac{t}{t+|x-y|}, & t>0,\\ 0, & t=0\end{cases}$$

and $$N(x,y,t)=\begin{cases}\dfrac{|x-y|}{t+|x-y|}, & t>0,\\ 1, & t=0\end{cases}.$$

Clearly, (X, M, N, *, ◊) is an intuitionistic fuzzy metric space.

Define g(x) = $\frac{x}{6}$ and f(x) = $\frac{x}{2}$ on X. It is clear that g(x) $\subseteq$ f(x).

Now, $$M\left(gx,gy,\frac{t}{3}\right)=\frac{\frac{t}{2}}{\frac{t}{2}+\frac{|x-y|}{6}}=\frac{3t}{3t+|x-y|}\geq\frac{t}{t+\frac{|x-y|}{3}}=\frac{3t}{3t+|x-y|}=M(fx,fy,t),$$

and

$$N\left(gx,gy,\frac{t}{3}\right)=\frac{\frac{|x-y|}{6}}{\frac{t}{2}+\frac{|x-y|}{6}}=\frac{|x-y|}{3t+|x-y|}\leq\frac{\frac{|x-y|}{3}}{t+\frac{|x-y|}{3}}=\frac{|x-y|}{3t+|x-y|}=N(fx,fy,t).$$

Thus, all the conditions of Theorem 3.1 are satisfied and so f and g have a unique common fixed point 0.

As an application of Theorem 3.1, we prove a common fixed point theorem for four finite families of mappings which runs as follows:

Theorem 3.2. Let $\{f_1, f_2, \ldots, f_m\}$ and $\{g_1, g_2, \ldots, g_n\}$ be two finite families of self-mappings of a intuitionistic fuzzy metric spaces with continuous t-norm * and continuous t-conorm ◊ defined by $a*a \geq a$ and $(1-a)\ \Diamond\ (1-a) \leq (1-a)$ for all $a \in [0, 1]$ such that $f = f_1f_2 \ldots f_m$, $g = g_1g_2 \ldots g_n$, satisfy condition (3.1), (3.2) and (3.3).

Then f and g have a point of coincidence. Moreover, if $f_if_j = f_jf_i$, $g_kg_l = g_lg_k$ for all $i, j \in I_1 = \{1, 2, \ldots, m\}$, $k, l \in I_2 = \{1, 2, \ldots, n\}$, then (for all $i \in I_1$, $k \in I_2$) f_i and g_k have a common fixed point.

Proof. The conclusion is immediate i.e., f and g have a point of coincidence as f and g satisfy all the conditions of Theorem 3.1. Now appealing to component wise commutativity of various pairs, one can immediately prove that fg = gf, hence, obviously pair (f, g) is sub-compatible. Note that all the conditions of Theorem 3.1 are satisfied which ensured the existence of a unique common fixed point, say z. Now one need to show that z remains the fixed point of all the component maps.

For this consider

$$f(f_i z) = ((f_1 f_2 \dots f_m) f_i) z = (f_1 f_2 \dots f_{m-1})((f_m f_i) z) = (f_1 \dots f_{m-1})(f_i f_m z)$$

$$= (f_1 \dots f_{m-2})(f_{m-1} f_i (f_m z) = (f_1 \dots f_{m-2})(f_i f_{m-1}(f_m z))$$

$$= \dots f_1 f_i (f_2 f_3 f_4 \dots f_m z) = f_i f_1 (f_2 f_3 \dots f_m z)$$

$$= f_i (fz) = f_i z.$$

Similarly, one can show that

$$f(g_k z) = g_k(fz) = g_k z, \; g(g_k z) = g_k(gz) = g_k z$$

and $g(f_i z) = f_i(gz) = f_i z,$

which show that (for all i and k) $f_i z$ and $g_k z$ are other fixed points of the pair (f, g).

Now appealing to the uniqueness of common fixed points of both pairs separately, we get

$z = f_i z = g_k z$, which shows that z is a common fixed point of f_i, g_k for all i and k.

Conclusion. Theorem 3.1 is a generalization of the result of Turkoglu et. al. [11] in the sense that condition of commuting mappings of the pairs of self maps has been restricted to sub-compatible sub-sequential self maps.

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